ROSSBY WAVES IN THE OCEAN (Part 1)

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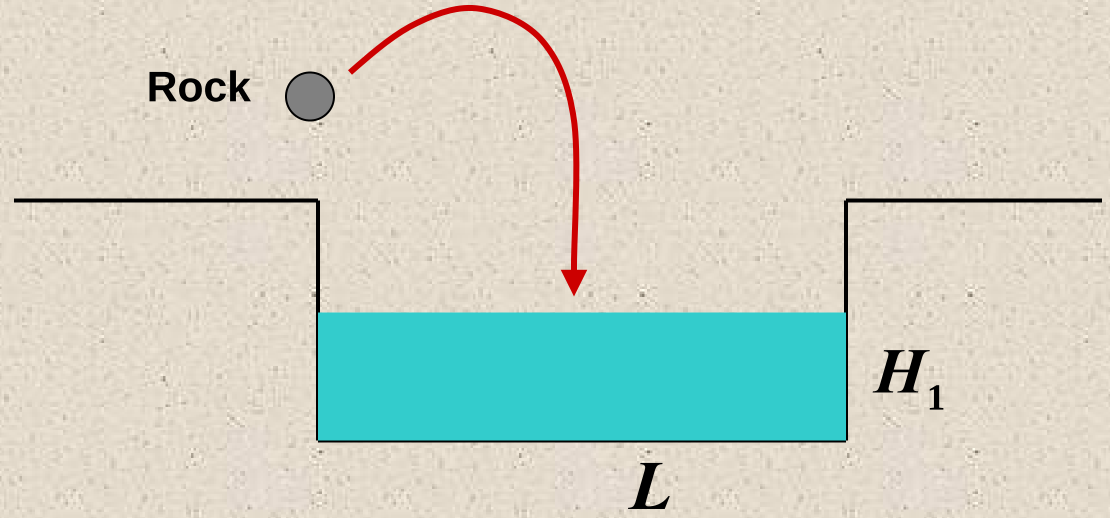
April 5, 2010

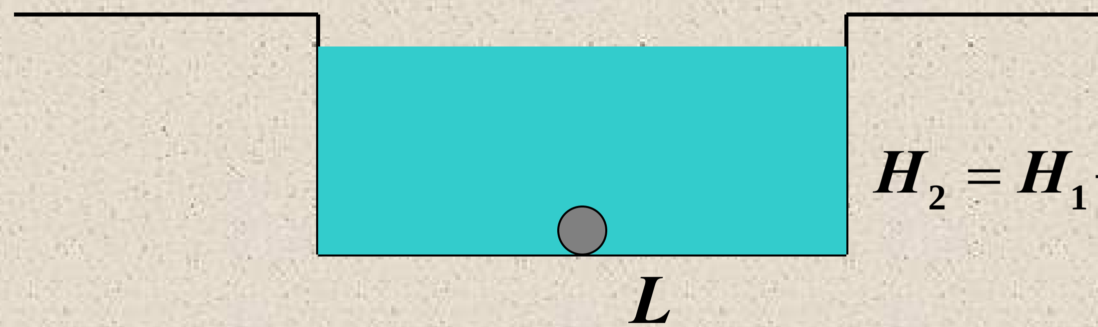
(1)



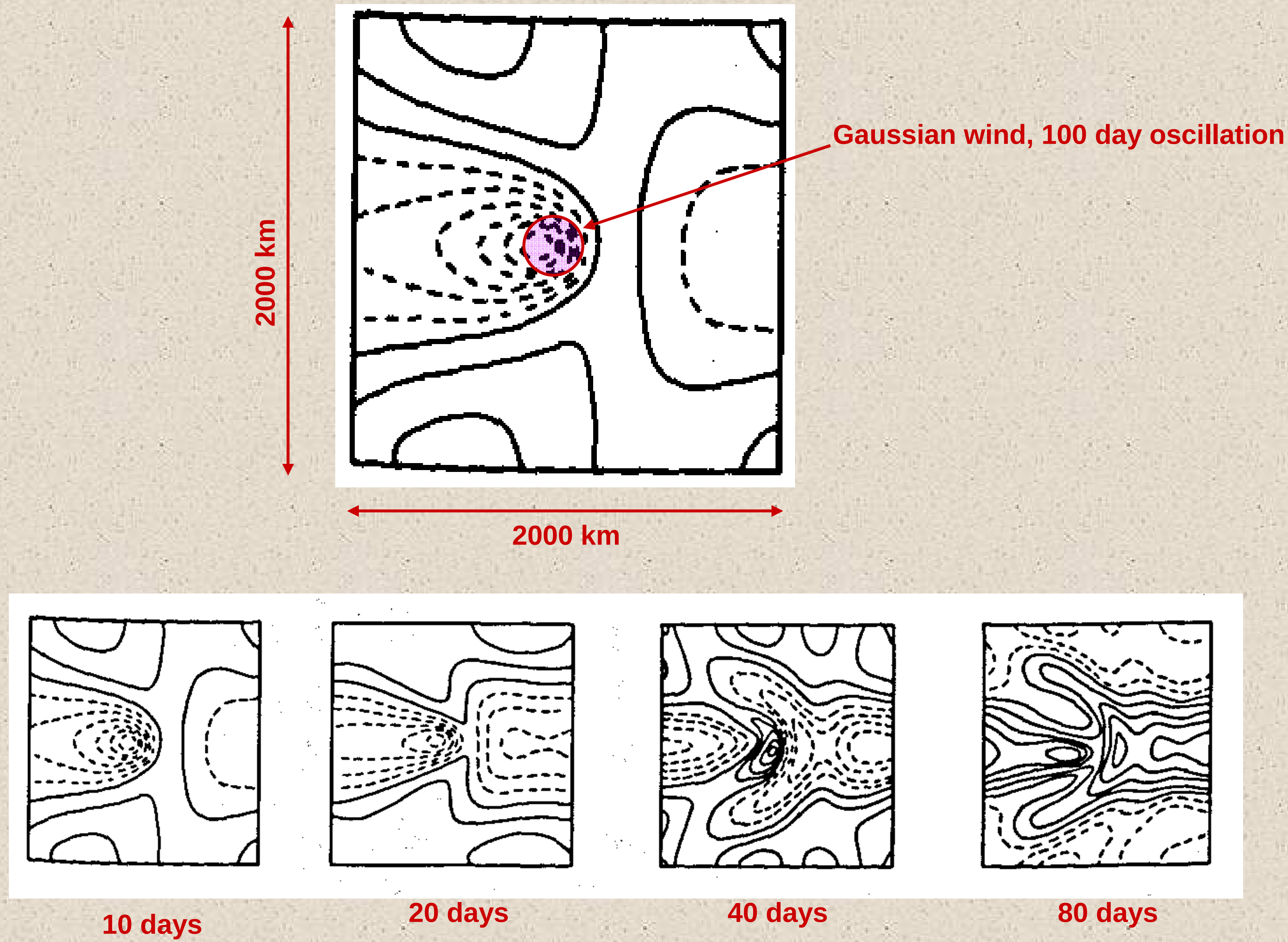
(+ waves)

$$c \approx \sqrt{g\,\lambda} \rightarrow T = \frac{L}{c} \qquad \lambda \leq \frac{L}{2}$$

(2)



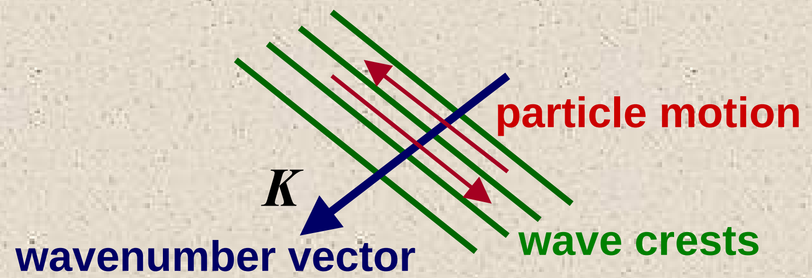
$$H_2 = H_1 + \left(\frac{\rho_{\text{rock}}}{\rho_{\text{water}}}\right)\left(\frac{V_{\text{rock}}}{A_{\text{pond}}}\right)$$



[Haidvogel and Rhines, 1983]

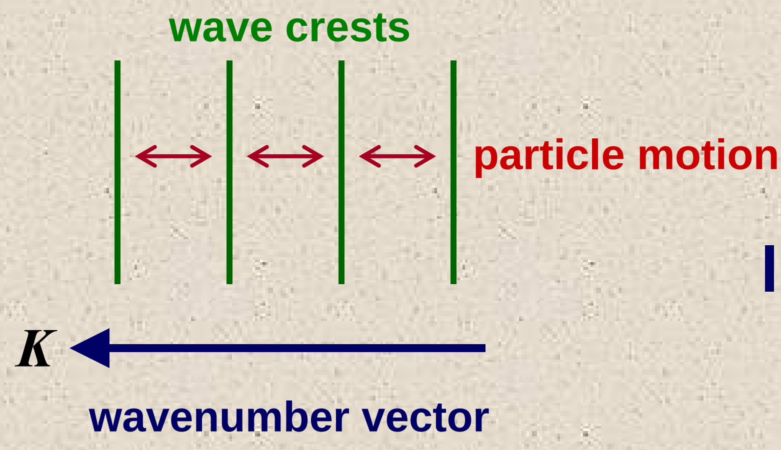
Transverse and Longitudinal Waves

$$\nabla \bullet \vec{\boldsymbol{u}} = 0\,, \quad \boldsymbol{u} \sim \mathrm{e}^{i(kx+ly+mz-\sigma t)} \Rightarrow \vec{\boldsymbol{u}} \bullet \boldsymbol{K} = 0 \qquad \boldsymbol{K} = (k, l, m)$$



transverse wave

(ocean waves)



longitudinal wave

(sound waves)

All linear wave problems will result in a governing PDE of the form

$$\Re(\varphi) = 0$$

($\Re$ is a linear differential operator)

The approach to solving this PDE will generally be to assume solutions of the form

$$\varphi \sim e^{i(kx + ly + mz - \sigma t)} \qquad \boldsymbol{K} = (k, l, m)$$

This substitution will yield a relationship between the frequency σ and wavenumber $\boldsymbol{K}$ in the form

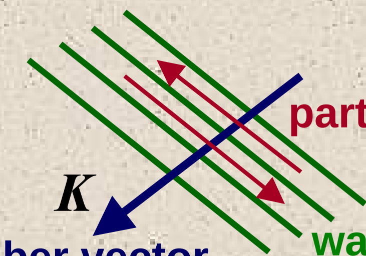
$$\sigma = \sigma(k, l, m) = \sigma(\boldsymbol{K})$$

$\boldsymbol{K}$ = wavenumber vector

$\rightarrow$ ***the dispersion relation***

Phase speed (NOT phase velocity)….

$$\nabla \bullet \vec{u} = 0\,, \quad \boldsymbol{u} \sim e^{i(kx+ly+mz-\sigma t)} \Rightarrow \vec{u} \bullet \boldsymbol{K} = 0 \qquad \boldsymbol{K} = (k, l, m)$$



Along the wavenumber vector, a line of constant phase moves at a speed given by

$$c = \frac{\sigma}{|\boldsymbol{K}|}$$

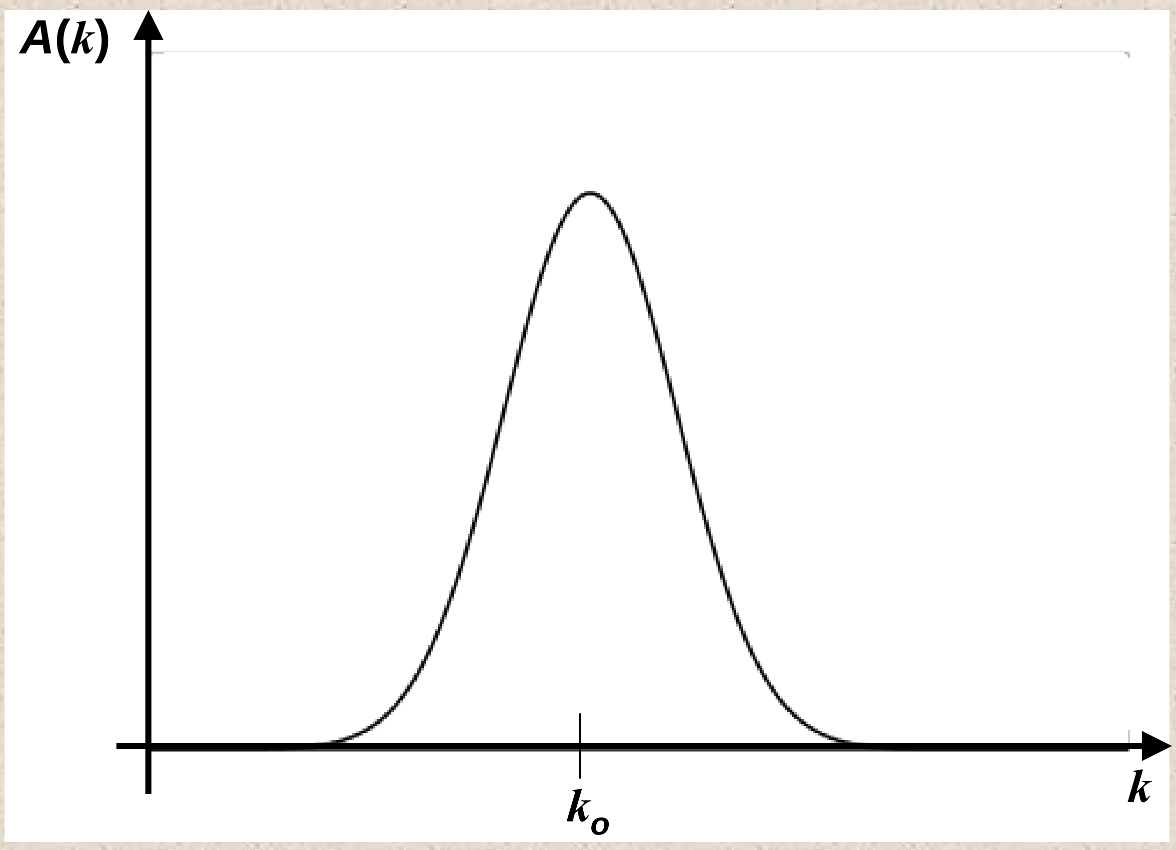
Note that this applies only *to a single, monochromatic wave*, and note that c is not a vector in the normal sense,

$$c \neq (c_x, c_y, c_z) \neq \left(\frac{\sigma}{k}, \frac{\sigma}{l}, \frac{\sigma}{m}\right) \qquad \boldsymbol{K} = (k, l, m) \rightarrow \frac{\boldsymbol{K}}{\sigma} = \left(\frac{k}{\sigma}, \frac{l}{\sigma}, \frac{m}{\sigma}\right)$$

“slowness”

A more useful description…..group velocity

Group velocity: the motion of a packet of waves



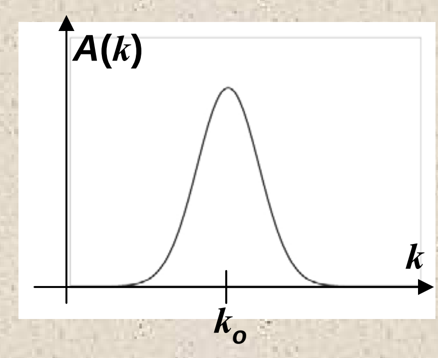
The distribution of amplitudes of a wave packet in k-space

Suppose there is a wave packet moving in the +*x* direction, which can be written as

$$\varphi(x,t)=\int A(k)\mathrm{e}^{ikx}\mathrm{e}^{-i\sigma(k)t}dk$$

and further suppose that *A*(*k*) is peaked around a wavenumber k_o.

In the vicinity of k_o the dispersion relation is



$$\sigma(k)=\sigma(k_o)+\frac{\partial\sigma}{\partial k}\Big|_{k_o}(k-k_o)+\ldots$$

so that

$$\varphi(x,t)=\int A(k)\mathrm{e}^{ikx}\,\mathrm{e}^{-i\left[\sigma(k_o)+\frac{\partial\sigma}{\partial\kappa}\big|_{k_o}(k-k_o)+\ldots\right]t}dk$$

$$=\int A(k)\mathrm{e}^{ik(x-\frac{\partial\sigma}{\partial\kappa}\big|_{k_o}t)}dk\;\;\mathrm{e}^{i\left[\frac{\partial\sigma}{\partial\kappa}\big|_{k_o}k_o-\sigma(k_o)\right]t}dk$$

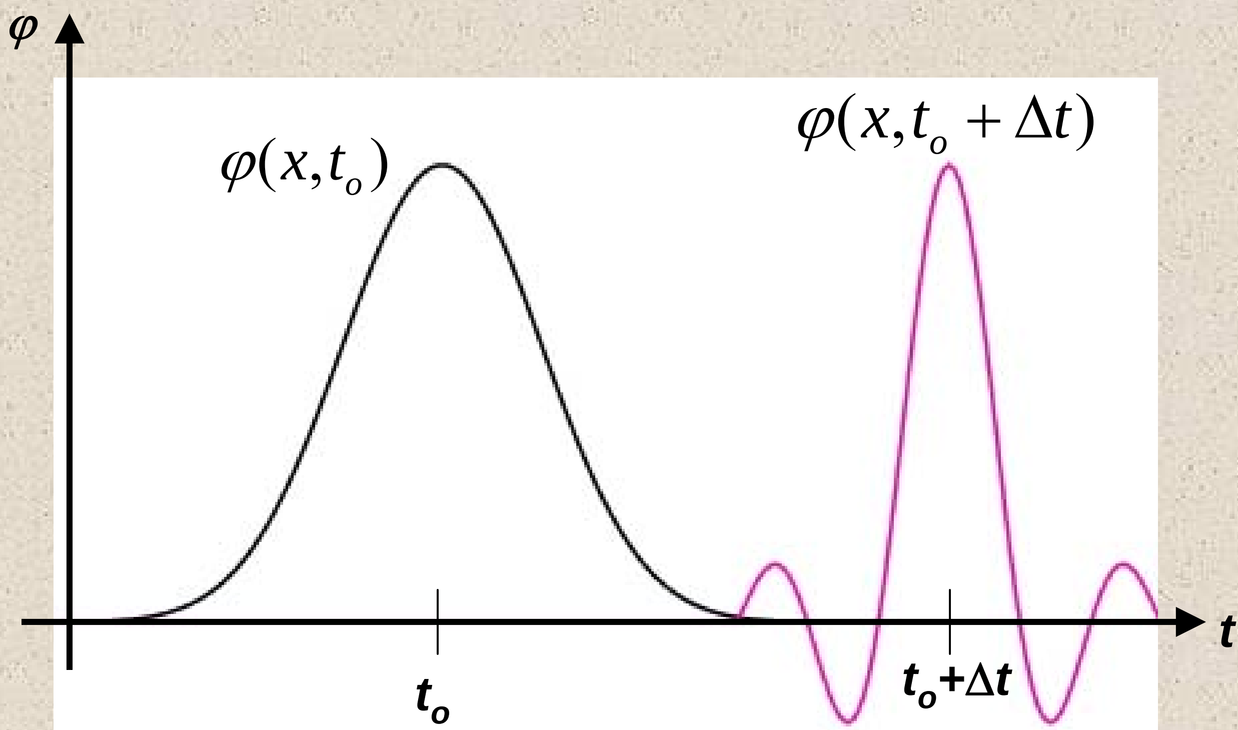
$$=\varphi\left[x-\frac{\partial\sigma}{\partial k}\Big|_{k_o}t,0\right]\mathrm{e}^{\iota\left[\frac{\partial\sigma}{\partial\kappa}\big|_{\kappa_o}\kappa_o-\sigma(k_o)\right]\tau}\quad=\varphi(x-c_{gx}t,0)\,\mathrm{e}^{i\left[\frac{\partial\sigma}{\partial\kappa}\big|_{\kappa_o}k_o-\sigma(k_o)\right]t}$$

$$\varphi(x,t) \approx \varphi(x - c_{gx}t, 0)\mathrm{e}^{i\left[\frac{\partial\sigma}{\partial\kappa}\Big|_{k_o} \boldsymbol{k_o} - \sigma(\boldsymbol{k_o})\right]\boldsymbol{t}}$$

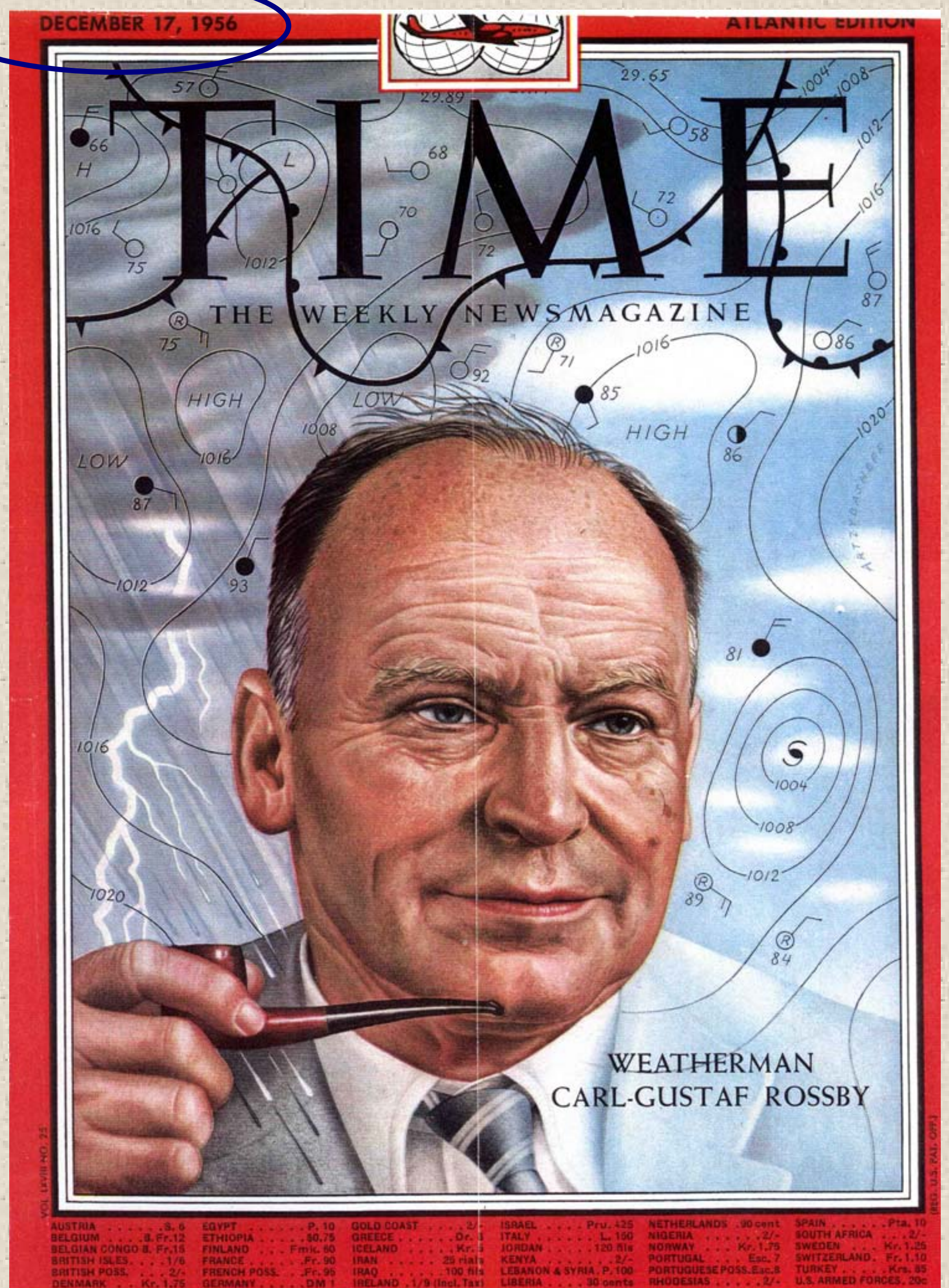
modulation of the original packet

translation of the original packet at speed c_{gx}

c_{gx} = the x-group velocity; the speed at which energy of the packet moves

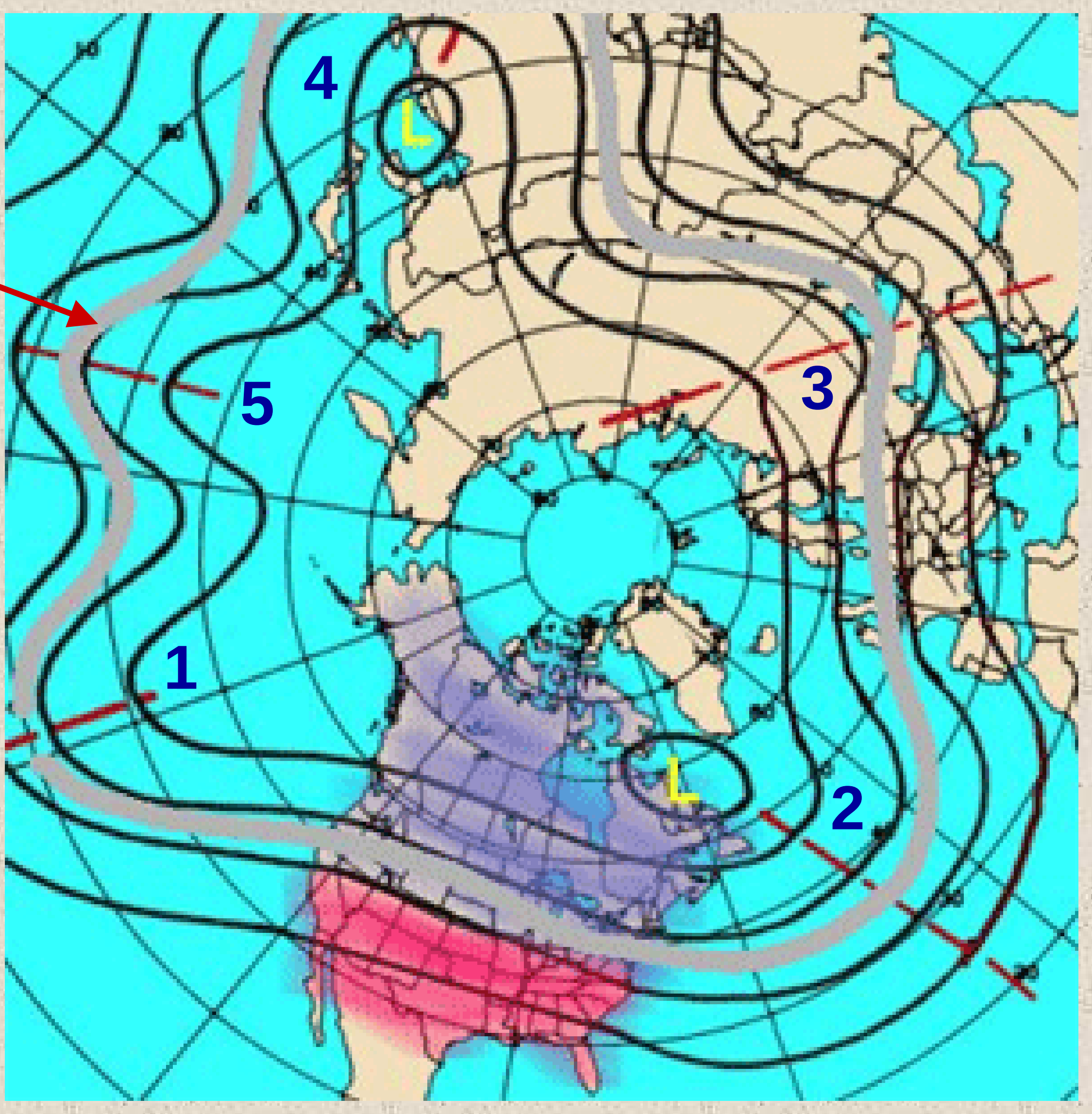


$$\varphi(x,t) \approx \varphi(x - c_{gx}t, 0)\mathrm{e}^{i\left[\left.\frac{\partial\sigma}{\partial\kappa}\right|_{k_o} k_o - \sigma(k_o)\right]t}$$



DECEMBER 17, 1956
ATLANTIC EDITION
TIME
THE WEEKLY NEWSMAGAZINE
WEATHERMAN
CARL-GUSTAF ROSSBY

Jet stream



Rossby waves in the atmosphere [500 mb pressure]

[what about the ocean?]

A Short History of Rossby Waves

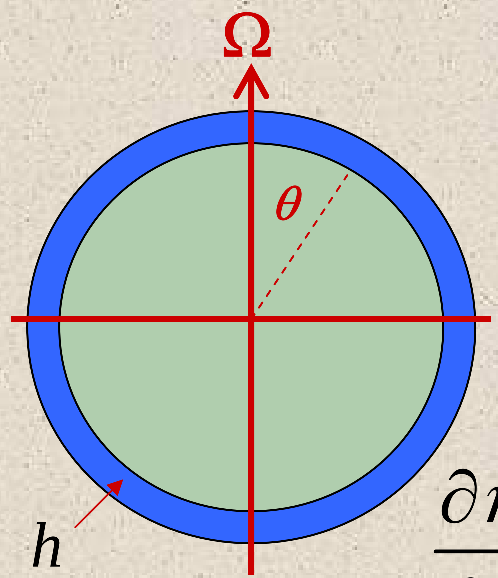
- **Laplace (1799) formulated and discussed his tidal equations and showed 2 types of solutions ("Motions of the First Class", and "Motions of the Second Class").**

- **Margules (1893) examined the free oscillations of a rotating, planetary atmosphere, confirming the existence of Laplace's second class solutions.**

- **Hough (1898) examined the free oscillations of a global ocean of uniform depth, again confirming the second class solutions, and found the eigenfunctions of the solution on a sphere.**

- **Rossby (1939) introduced the β-plane approximation and was able to study the transformed Laplace tidal equations in detail for the atmosphere, resulting in important advances in the study of Laplace's motions of the second class.**

Laplace's Tidal Equations (LTE)



$$\frac{\partial u}{\partial t} - 2\Omega v\cos\theta = -\frac{g}{a\sin\theta}\frac{\partial \eta}{\partial \phi}$$

$$\frac{\partial v}{\partial t} + 2\Omega u\sin\theta = \frac{g}{a}\frac{\partial \eta}{\partial \theta}$$

$$\frac{\partial \eta}{\partial t} + \frac{h}{a\sin\theta}\left[\frac{\partial}{\partial \theta}\left(-v\sin\theta\right) + \frac{\partial u}{\partial \phi}\right] = 0$$

$h =$ mean ocean depth $\quad \eta =$ perturbed sea level

$\mathbf{u} = (u, v, w) \quad a =$ Earth radius

$\theta = \frac{\pi}{2} -$ latitude; $\quad \phi =$ longitude; $\quad \Omega =$ rotation rate

The LTE can be combined to form a single PDE in terms of the variable η (sea level):

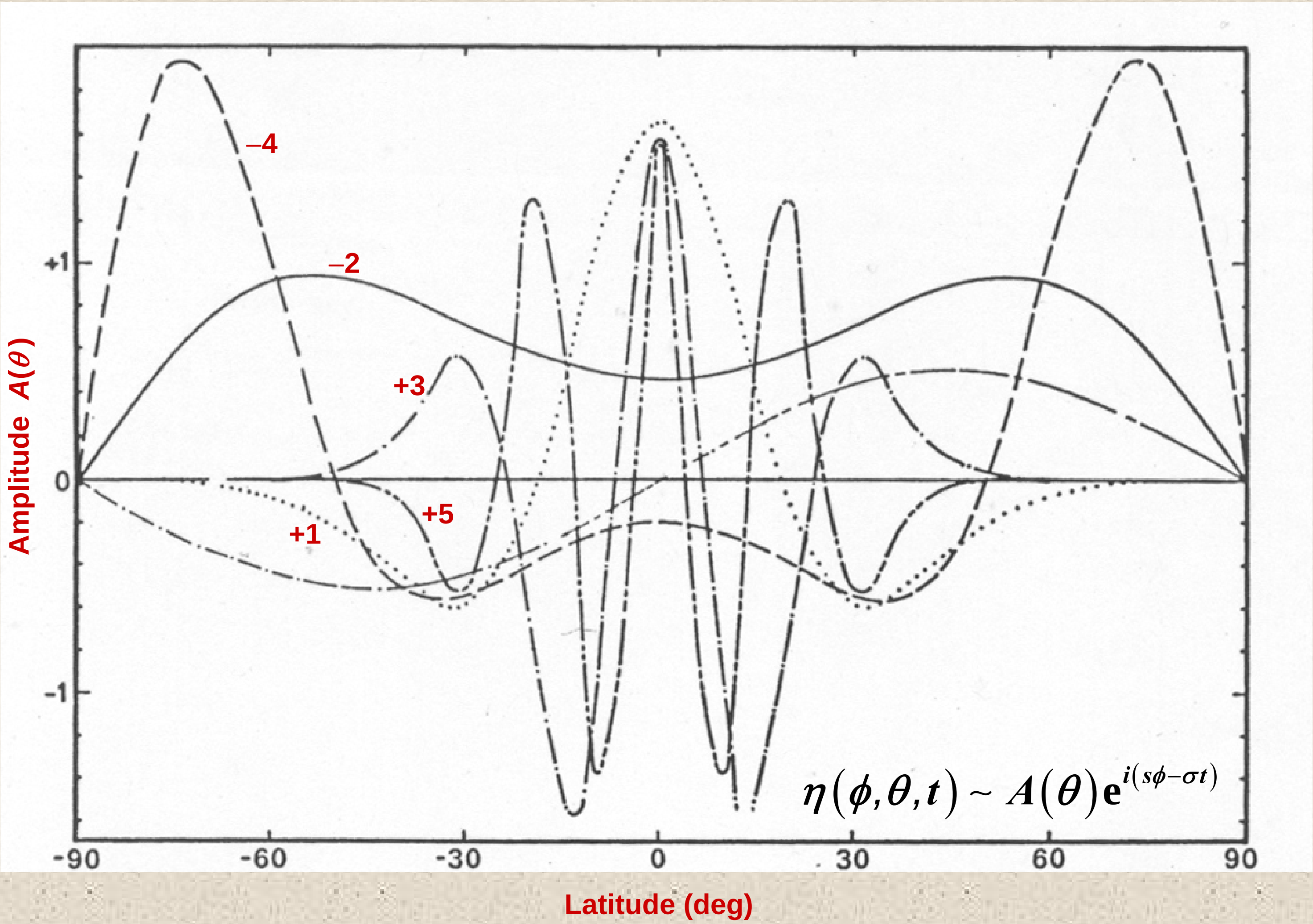
Put $\boldsymbol{\eta} \sim \mathrm{e}^{i(s\phi - \sigma t)}$ $\left(\text{where } s \text{ is any integer}\right)$ to find

$$\Re\left(\boldsymbol{\eta}\right) = \varepsilon \boldsymbol{\eta} \quad \left(\text{where } \Re \text{ is a linear operator}\right) \qquad \varepsilon = \frac{4\Omega^2}{gh} \qquad \gamma = \frac{\boldsymbol{\sigma}}{2\Omega}$$

$$\Re = \frac{1}{\gamma \sin\boldsymbol{\theta}}\left[\frac{\partial}{\partial\boldsymbol{\theta}}\left\{\frac{1}{\gamma^2 - \cos^2\boldsymbol{\theta}}\left(s\cos\boldsymbol{\theta} - \gamma\sin\boldsymbol{\theta}\frac{\partial}{\partial\boldsymbol{\theta}}\right)\right\} + \frac{s}{\gamma^2 - \cos^2\boldsymbol{\theta}}\left(\frac{\gamma s}{\sin\boldsymbol{\theta}} - \cos\boldsymbol{\theta}\frac{\partial}{\partial\boldsymbol{\theta}}\right)\right]$$

This is a formidable PDE to solve; it is not generally separable in θ and λ. Approximate solutions have been found by Margules, Hough, Longuet-Higgins, and others.

Hough's approximate solution: eigenfunctions on of the LTE on a sphere in terms of Legendre polynomials (later called *Hough functions*).

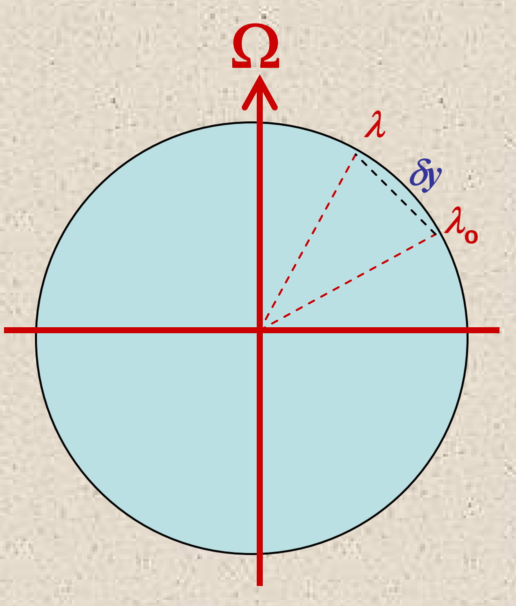


Rossby's Contribution: The β - plane

$$f = 2\Omega \sin\lambda \quad ; \quad \lambda = \text{latitude} \quad ; \quad \theta = \frac{\pi}{2} - \lambda$$

$$f - f_o = 2\Omega\left(\sin\lambda - \sin\lambda_o\right) \approx 2\Omega\left(\sin\lambda_o + [\lambda - \lambda_o]\cos\lambda_o + \ldots\ldots - \sin\lambda_o\right)$$

$$f - f_o = 2\Omega\cos\lambda_o[\lambda - \lambda_o] = 2\Omega\cos\lambda_o\left(\frac{\delta Y}{R}\right)$$



$$\underline{f = f_o + \beta y} \quad ; \quad \beta = \frac{2\Omega\cos\lambda_o}{R} \quad ; \quad y = \delta Y$$

the β– plane approximation; replaces spherical coordinates with Cartesian coordinates

[see Veronis (1963) for a complete treatment]

The β – plane

$$f = f_o + \beta y \quad ; \quad \beta = \frac{2\Omega \cos\lambda_o}{R} \quad ; \quad y = \delta Y$$

Note that only the first term of the Taylor series in λ has been kept; this implies that βy << f_o for consistency.

[unless f_o = 0]

Thus,

$$\frac{\beta y}{f_o} << 1 \quad \Rightarrow \quad \left(\frac{2\Omega \cos\lambda_o}{R}\right)\frac{y}{\left(2\Omega \sin\lambda_o\right)} << 1 \quad \Rightarrow \quad y << R \tan\lambda_o$$

At a latitude of 20°, for example,

$$y << R \tan\lambda_o = \left(6.3 \times 10^3\right)\left(0.36\right) = \underline{2.3 \times 10^3 \text{ km}}$$

Use of the β – plane

$$\frac{\partial u}{\partial t} - fv = -g\frac{\partial \eta}{\partial x}$$

$$\frac{\partial v}{\partial t} + fu = -g\frac{\partial \eta}{\partial y}$$

u **and** ***v*** **momentum equations**

$$f = f_o + \beta y$$

$$\frac{\partial \eta}{\partial t} + \frac{\partial}{\partial x}(hu) + \frac{\partial}{\partial y}(hv) = 0$$

integrated continuity equation

Put $u, v, \eta \sim \mathrm{e}^{-i\sigma t}$

These can be combined into a single PDE for *v* :

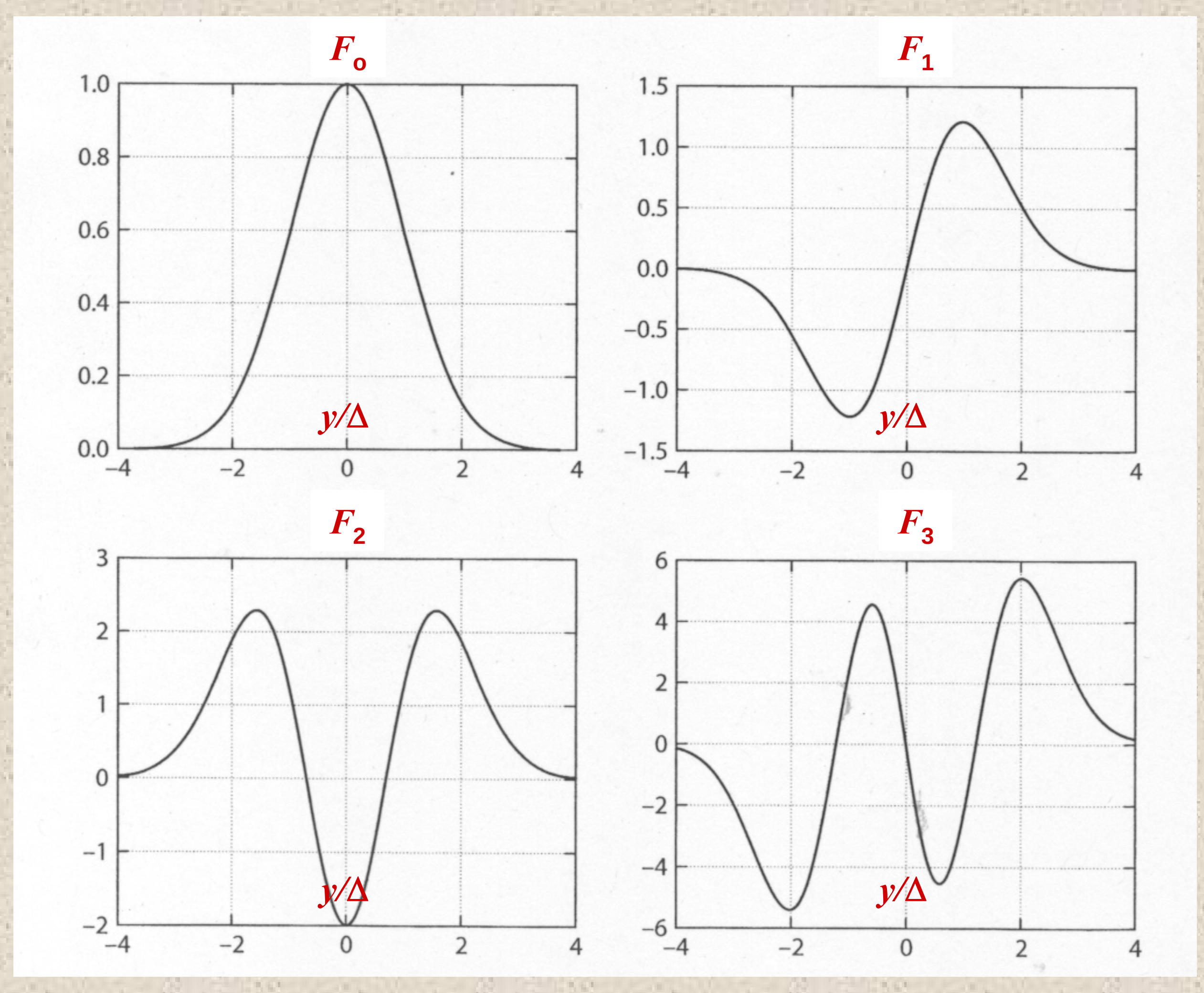
$$\nabla^2 v + \left(\frac{i\beta}{\sigma}\right)\frac{\partial v}{\partial x} + \frac{\sigma^2 - f^2}{gh} v = 0 \qquad f = f_o + \beta y$$

Case (1): near the Equator $f_o = 0$, so that $f = \beta y$.

put $v = e^{-i\left(\frac{\beta}{2\sigma} + kx\right)} F(y)$ **(separation of variables)**

$$\frac{d^2 F}{dy^2} + \left\{\left[\frac{\sigma^2}{gh} - \frac{\beta^2}{4\sigma^2} - k^2\right] - \frac{\beta^2 y^2}{gh}\right\} F = 0$$

[a parabolic cylinder ODE; solution are Hermite functions]



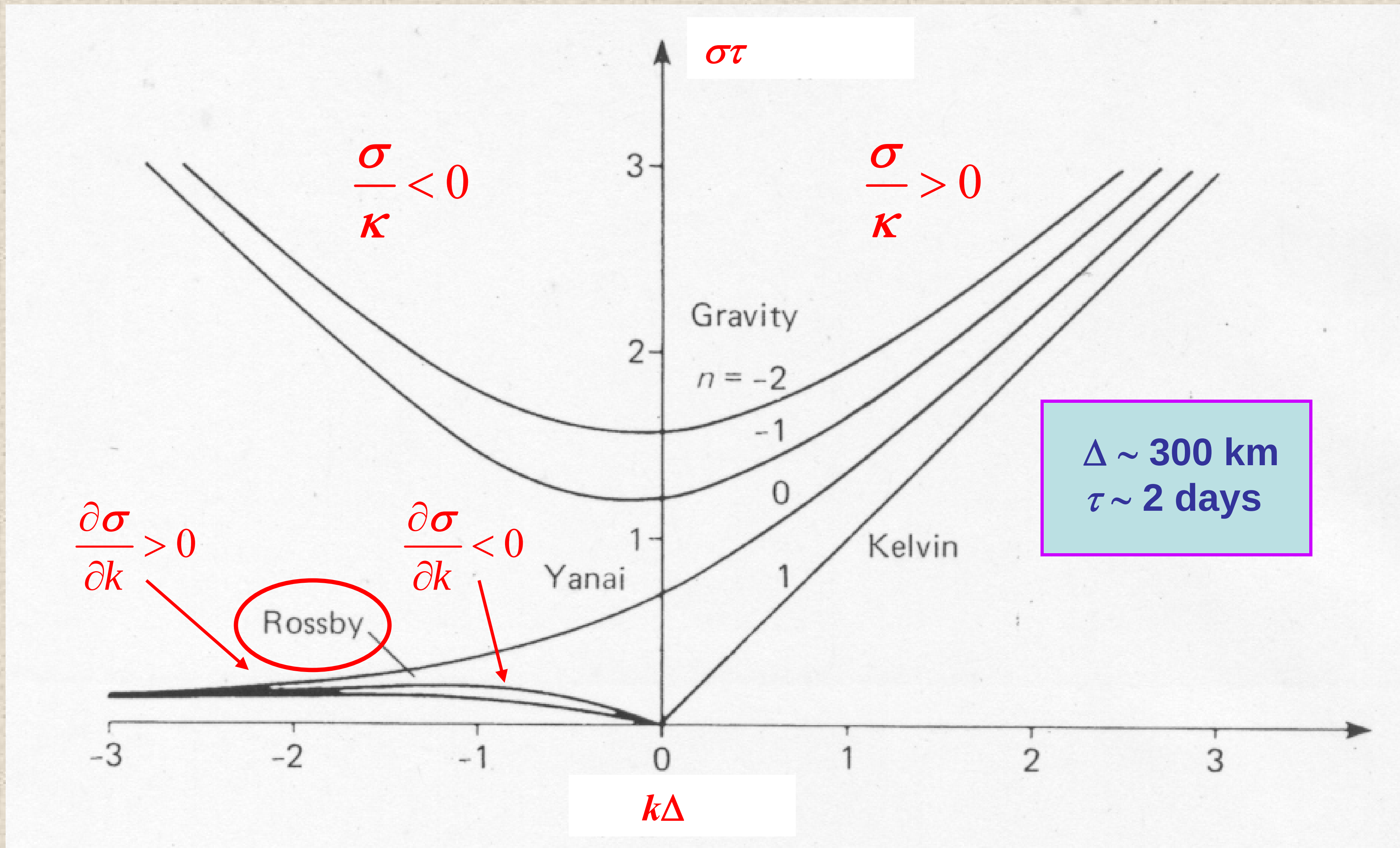
The first few Hermite functions, showing the meridional structure of equatorially-trapped waves (*note*: $\Delta \sim 300$ km)

The separated ODE yields the dispersion relation

$$\frac{\sqrt{gh}}{\beta}\left(\frac{\sigma^2}{gh}-k^2-\frac{\beta k}{\sigma}\right)=2n+1$$

n **= 0, 1, 2, ……**

which is cubic in σ and thus has 3 roots for σ for any choice of n .



The dispersion relation for near-Equatorial waves, determined from the eigenvalue ODE; note that there are several distinct types of waves.

Case (2): away from the Equator, so that $f = f_o + \beta y$.

$$\frac{\partial u}{\partial t} - fv = -\frac{1}{\rho}\frac{\partial p}{\partial x}$$

$$\frac{\partial v}{\partial t} + fu = -\frac{1}{\rho}\frac{\partial p}{\partial y}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

The Rigid Lid Approximation

$$\frac{\partial p}{\partial z} = -g\,\rho \;\Rightarrow\; p(z) = -g\,\rho(z-\zeta)$$

$$= g\,\rho\zeta \text{ at } z = 0 \;\Rightarrow\; \frac{1}{\rho}\frac{\partial p}{\partial z} = g\frac{\partial \zeta}{\partial z}$$

The same basic equations, with a rigid lid approximation.

Put $u, v, p \sim \mathrm{e}^{-i\sigma t}$

$$\frac{\partial u}{\partial t} \boxed{- fv = -\frac{1}{\rho}\frac{\partial p}{\partial x}}$$

$$\frac{\partial v}{\partial t} \boxed{+ fu = -\frac{1}{\rho}\frac{\partial p}{\partial y}}$$

geostrophic balance: generally accurate to within 1-2% error at mid-latitudes on *time scales > a few days* and *length scales > 10s of kilometers*

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

note: 3 equations, 3 unknowns
[resulting PDE admits a variety of solutions]

However, using the geostrophic approximation, we can make an initial estimate of the solution:

$$u \approx -\frac{1}{\rho f_o}\frac{\partial p}{\partial y} \qquad v \approx \frac{1}{\rho f_o}\frac{\partial p}{\partial x}$$

[geostrophic balance: a filter]

$$u \approx -\frac{1}{\rho f_o}\frac{\partial p}{\partial y} \qquad v \approx \frac{1}{\rho f_o}\frac{\partial p}{\partial x}$$

[like a streamfunction]

$$u = -\frac{\partial \psi}{\partial y} \qquad v = \frac{\partial \psi}{\partial x} \qquad \psi = \frac{p}{\rho f_o}$$

Cross-differentiate the u and v equations, substitute for the streamfunction, and put the results in the continuity equation to yield single PDE for ψ:

$$\frac{\partial}{\partial t}\nabla^2\psi + \beta\psi_x = 0$$

the Rossby wave equation

[a vorticity equation]

$$\frac{\partial}{\partial t}\nabla^2\psi + \beta\psi_x = 0$$

$$\frac{\partial}{\partial t}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial t}\left(\psi_{xx} + \psi_{yy}\right) = \frac{\partial}{\partial t}\nabla^2\psi$$

$$= \frac{\partial \zeta}{\partial t}$$

rate of change of relative vorticity

$$\beta\psi_x = \beta v = \frac{\partial}{\partial t}\left(\beta y\right) = \beta\frac{\partial y}{\partial t} = \frac{\partial f}{\partial t}$$

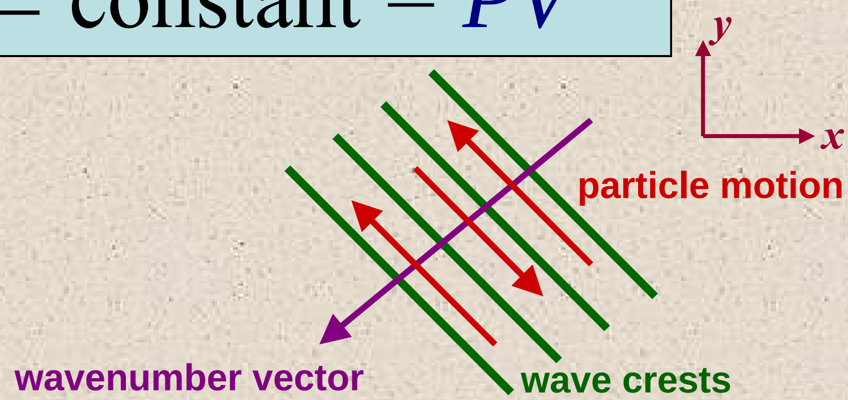
rate of change of planetary vorticity

$$\frac{\partial}{\partial t}\left[\nabla^2\psi + \beta y\right] = 0$$

$$\Rightarrow \nabla^2\psi + \beta y = \zeta + f = \text{constant} = PV$$

Assume wavelike solutions of the form

$$\psi = \mathrm{e}^{i\left(kx + ly - \sigma t\right)}$$



transverse waves

This substitution yields

$$-i\sigma\left(k^2 + l^2\right) + i\beta k = 0$$

which becomes the dispersion relation for Rossby waves,

$$\sigma = -\frac{\beta k}{k^2 + l^2}$$

Note the east-west phase speed:

$$\frac{\sigma}{k} = -\frac{\beta}{k^2 + l^2} < 0$$

A single Rossby wave always propagates to the west (eastward propagation not allowed) !

Take $k \sim 2\pi$ /1000 km, l = 0, then $\sigma \sim 2\pi$ /(44 days) $\rightarrow \sigma / k \sim$ 40 cm/sec . For the North Pacific, about 10^4 km wide, this wave would require about 229 days to cross the basin.

$$\sigma = -\frac{\beta k}{k^2 + l^2}$$

Rossby wave dispersion relation

***Notes*:**

(1) The longest waves (k, $l \to 0$) have the highest temporal frequencies, $\sigma \to \infty$.

(2) Thus, to observe high-frequency waves, observations must be made over large spatial scales; *or,*

(3) To observe short waves, observations must be collected over a long time.

Taken together, (1)–(3) suggest it will be difficult to observe these waves in the ocean.

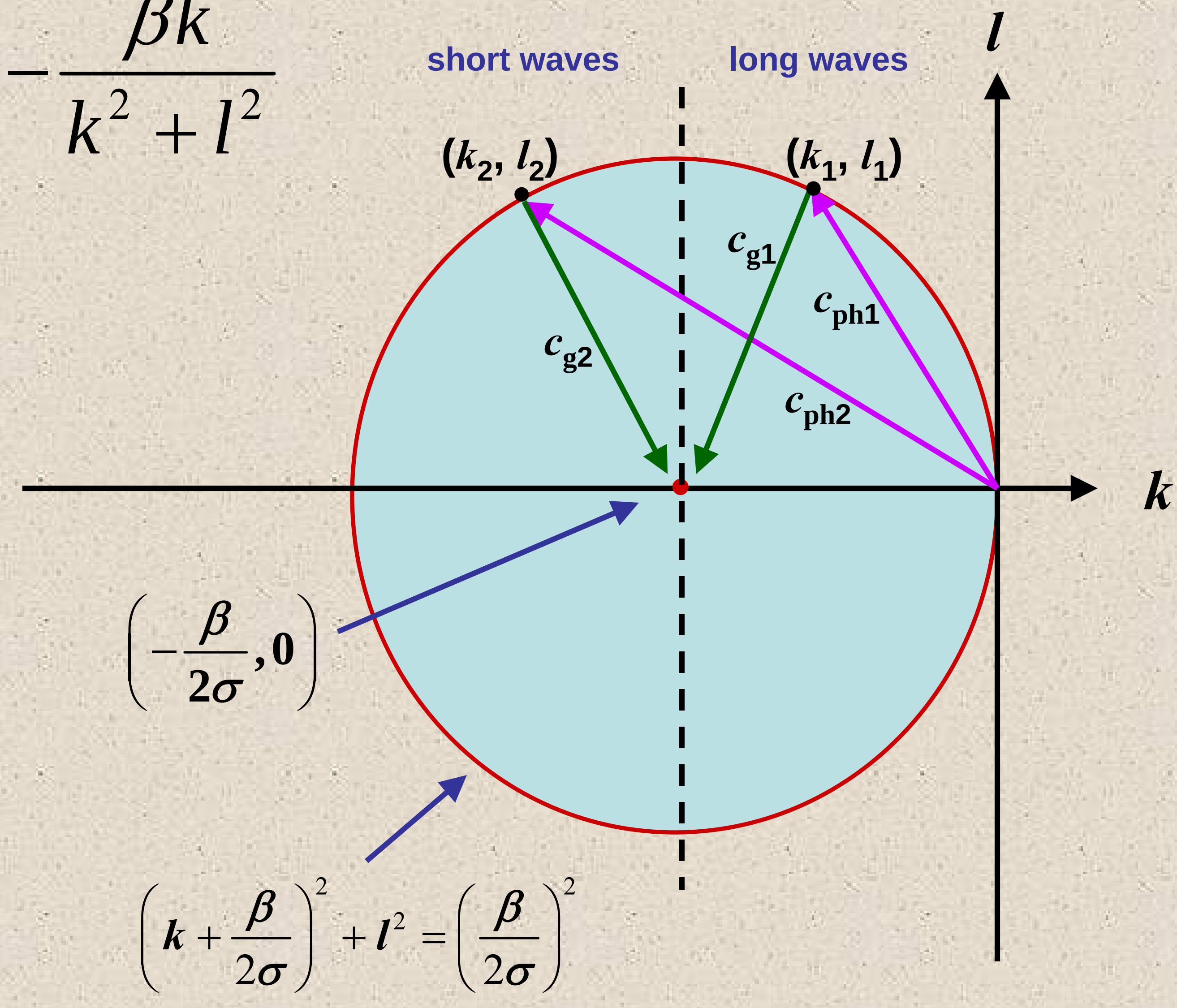
Group velocity….

$$\boldsymbol{c}_{\mathbf{g}} = \left(\frac{\partial \sigma}{\partial k}, \frac{\partial \sigma}{\partial l} \right)$$

$$\boldsymbol{c}_{\mathbf{g}} = \left(\frac{\beta\left(k^2 - l^2\right)}{\left(k^2 + l^2\right)^2}, \frac{2\beta k l}{\left(k^2 + l^2\right)^2} \right)$$

Result: the east-west group velocity of a packet of Rossby waves can be either in the *east or the west* direction, but the east-west phase speed for a single wave can be *only in the westward* direction !

$$\sigma = -\frac{\beta k}{k^2 + l^2}$$



Rossby Wave Dispersion Diagram

ROSSBY WAVES IN THE OCEAN (Part 2)

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Presented at National Institute of Oceanography, Goa, April 5, 2010

Effects of stratification….

$$\sim f\frac{\partial w}{\partial z}$$

$$\frac{\partial}{\partial t}\left[\nabla^2\psi+\frac{1}{f_o^2}\frac{\partial}{\partial z}\left(\frac{1}{N^2}\frac{\partial\psi}{\partial z}\right)\right]+\beta\psi_x=0$$

vortex stretching term

$$\rho=\rho_{oo}+\rho_o\left(z\right)+\rho'\left(x,y,z,t\right)\Rightarrow\rho'=-\frac{\rho_{oo}f_o}{g}\psi_z$$

Boussinesq approximation

$$\boxed{N^2=-\frac{g}{\rho_{oo}}\frac{\partial\rho_o}{\partial z}}$$

Look for solutions of the form

$$\psi=\phi\left(z\right)\mathrm{e}^{i\left(kx+ly-\sigma t\right)}$$

$$\psi = \phi(z)\mathrm{e}^{i(kx+ly-\sigma t)}$$

$$\longrightarrow \quad \phi(z) = \cos mz\ ,\ m = n\pi / H$$

solution for N = constant, with boundary conditions that w = 0 at the sea surface and bottom

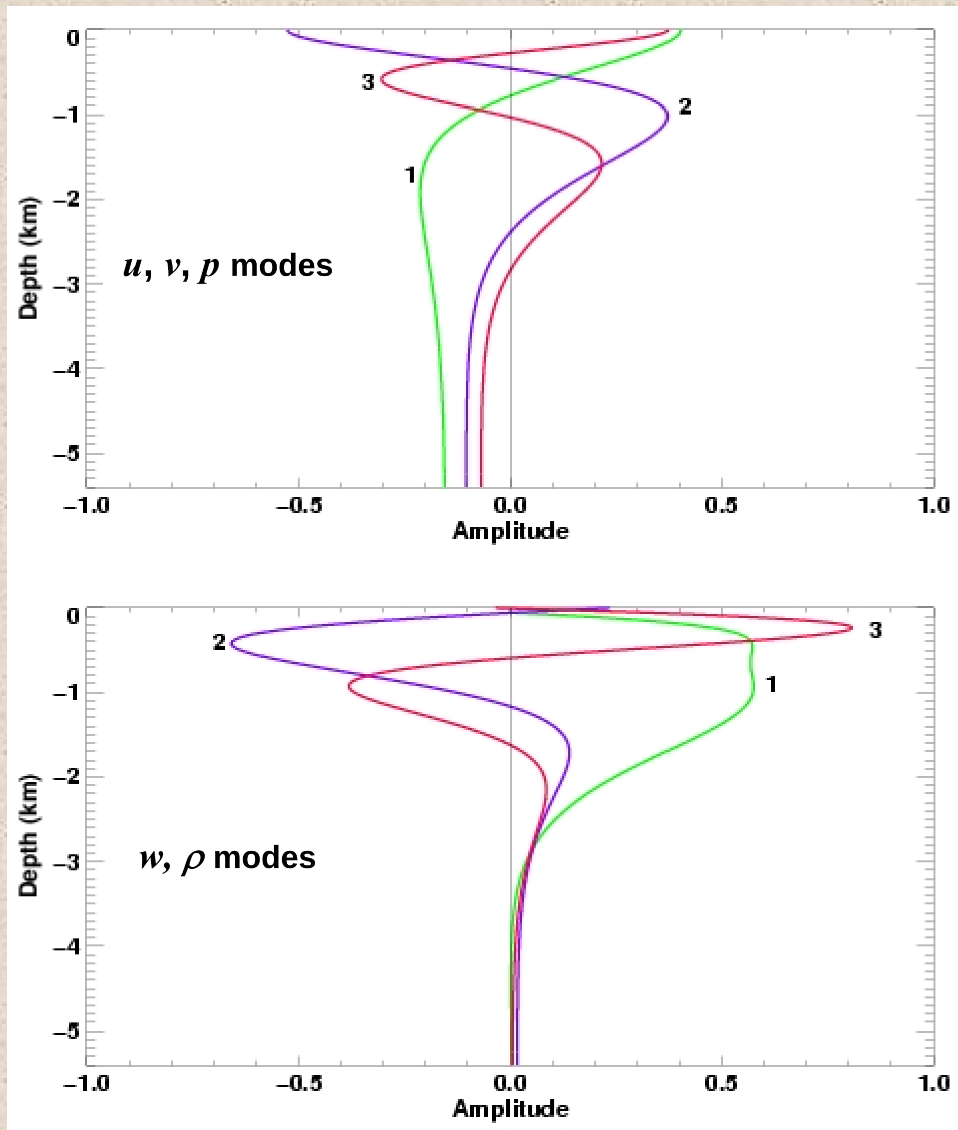
For a more realistic solution, take $N = N_o e^{az}$ (exponential stratification),

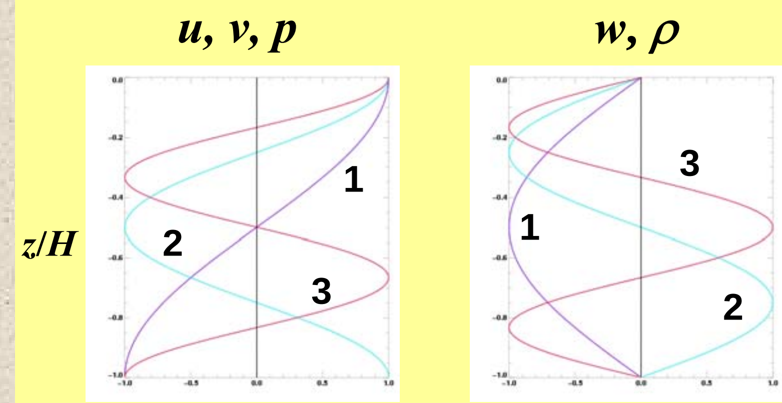
$$\phi(z) = \mathrm{e}^{az}\left[Y_1\left(\frac{c_n}{a}\mathrm{e}^{az}\right) - \frac{Y_o\left(\frac{c_n}{a}\right)}{J_o\left(\frac{c_n}{a}\right)} J_1\left(\frac{c_n}{a}\right)\mathrm{e}^{az} \right]$$

$\rightarrow$ **Y, J = Bessel functions, c_n = eigenvalues**

Vertical modal structure….

[exponential $N(z)$]





[constant $N(z)$]

$$\phi(z) = e^{az}\left[Y_1\left(\frac{c_n}{a}e^{az}\right) - \frac{Y_o\left(\frac{c_n}{a}\right)}{J_o\left(\frac{c_n}{a}\right)} J_1\left(\frac{c_n}{a}\right)e^{az} \right]$$

$$N = N_o e^{az} \quad , \quad a = 0.693\ \mathrm{km}^{-1}, \quad N_o = 6.04\times10^2\ \mathrm{day}^{-1}$$

$$c_n = (2.014,\ 4.279,\ 6.532)$$

$$H = 5.4\ \mathrm{km}$$

Dispersion relation....

$$\sigma_n = -\frac{\beta k}{k^2 + l^2 + \dfrac{f_o^2}{N_o^2} c_n^2}$$

For a given k and l, the baroclinic Rossby waves have a longer period than the barotropic waves

Take $k \sim 2\pi$/500 km, l = 0, n =1, then $\sigma \sim 2\pi$/(370 days) ; $\sigma/k \sim$ 1.3 cm/sec

For the North Pacific, approximately 10^4 km wide, this wave would take about 20 years to cross the basin.

Thus, baroclinic Rossby waves might travel very slowly (compared to barotropic waves) and would seem to be inherently nonlinear; this calls into question their possible existence.

The linear wave assumption….

$$\frac{|\mathrm{u}\bullet\nabla u|}{\dfrac{\partial u}{\partial t}} \sim \frac{\left(U^2 / L\right)}{\sigma U} << 1 \quad \Rightarrow \quad \frac{U}{\sigma L} << 1$$

neglected nonlinear terms

$$\Rightarrow \quad U << \sigma L \sim \frac{\sigma}{|K|}$$

particle speed

phase speed

For a linear wave, the speed of the fluid particles in the wave has to be much less than the wave phase speed; otherwise the wave is nonlinear and likely unstable.

Other important issues related to simple Rossby waves:

- **Effects of topography**

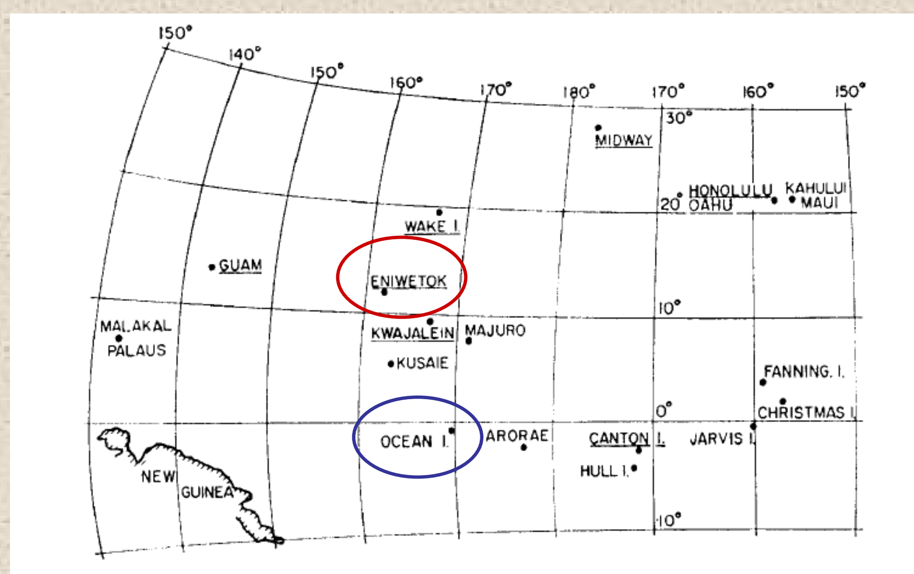
$$\frac{\beta y}{f_o} << 1 \; ; \qquad \text{suppose} \quad \frac{\beta y}{f_o} \sim \frac{\delta h}{h} = 0.01 \quad \Rightarrow$$

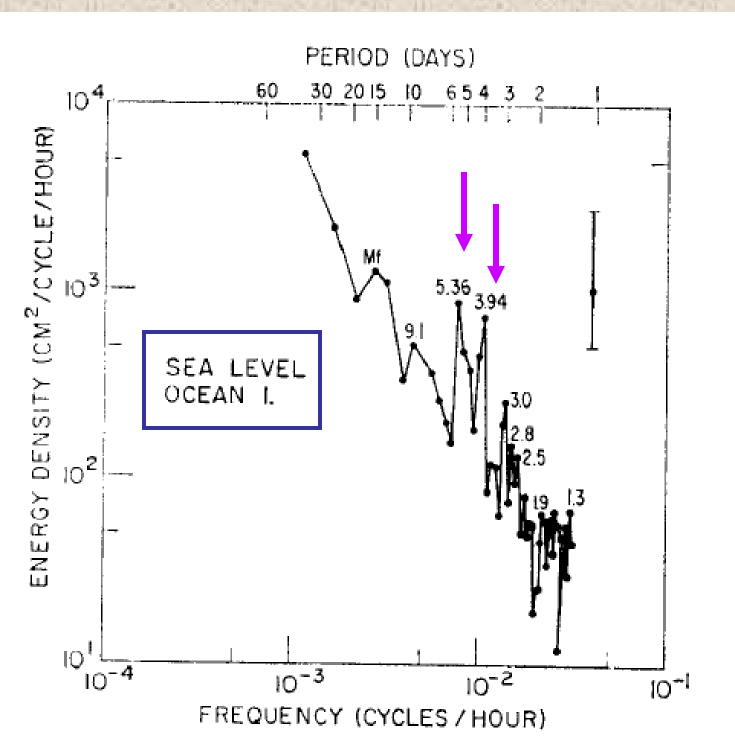
In a 5000 m ocean at 30°N, a change in depth of 50 meters over a N/S distance of ~ 85 km is the same size as the β-effect in the vorticity equation.

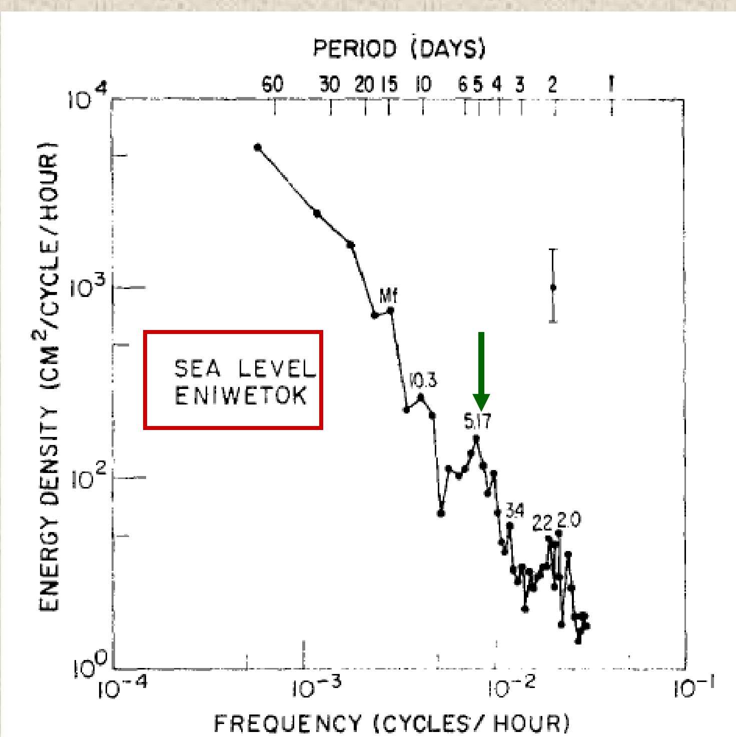
- **Wave-mean flow interactions**
- **Effects of coastlines (reflection properties)**
- **Nonlinearities**

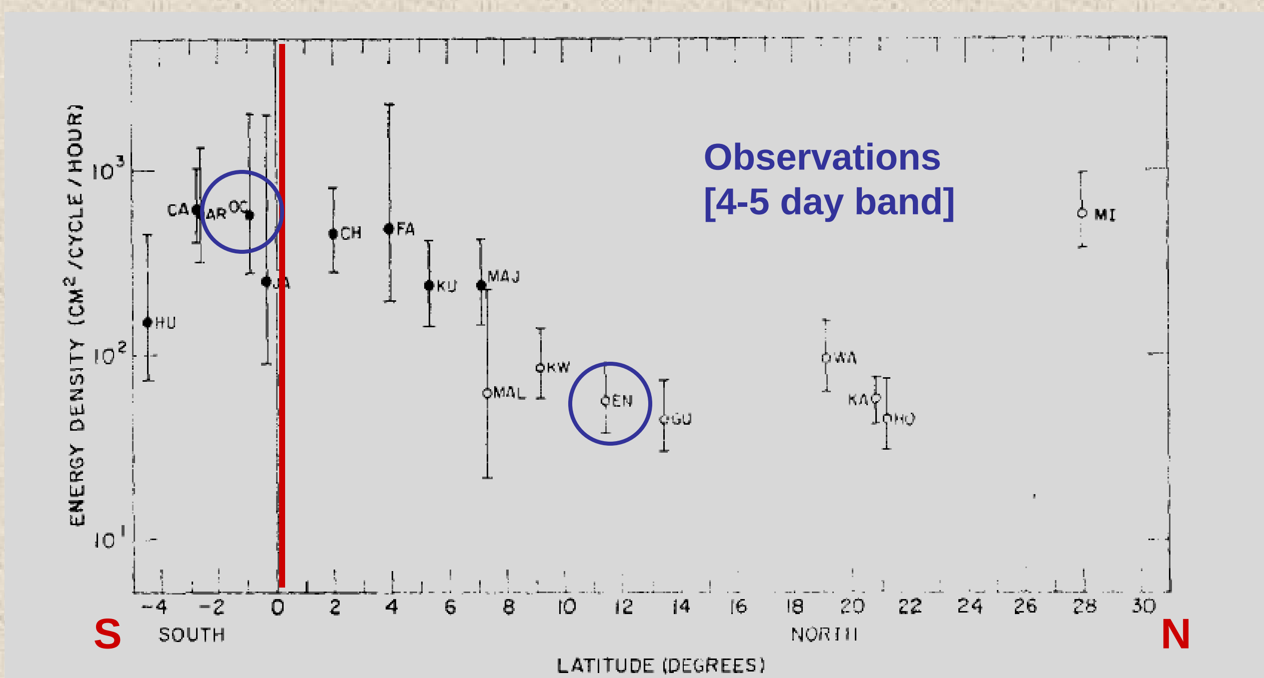
Pacific Island tide gauge stations

[Wunsch and Gill, 1976]

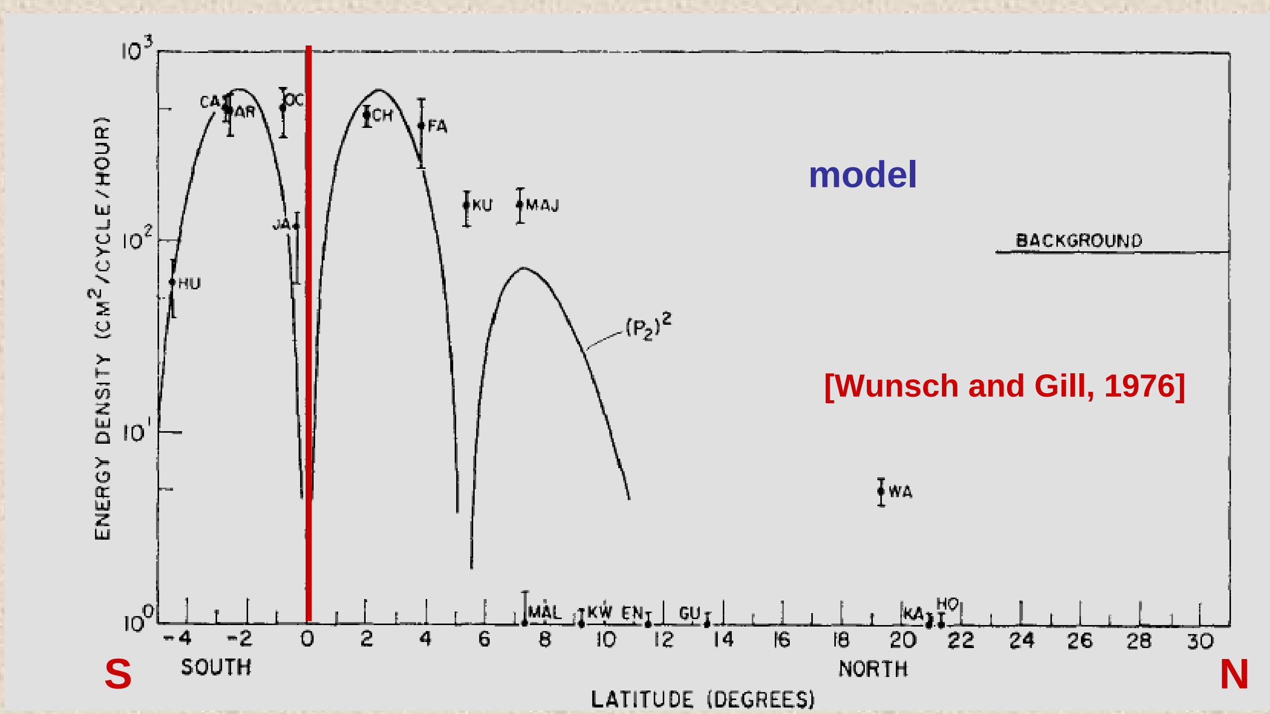




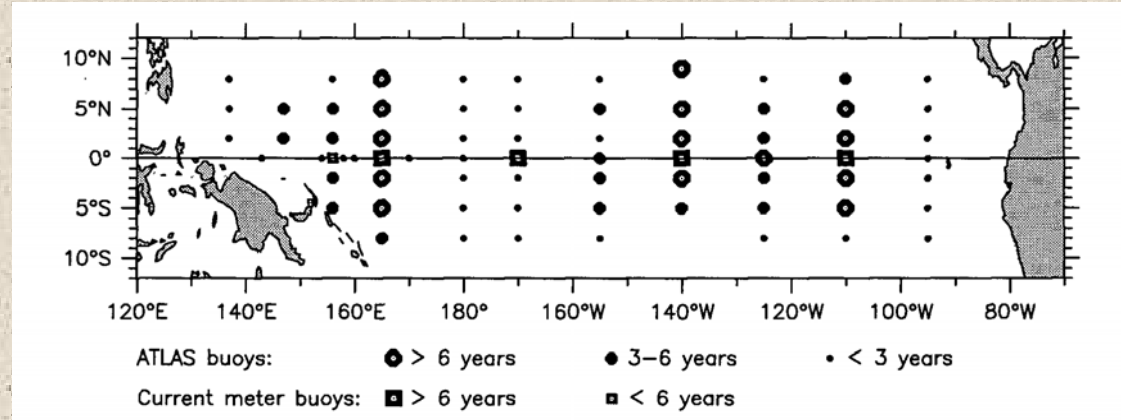




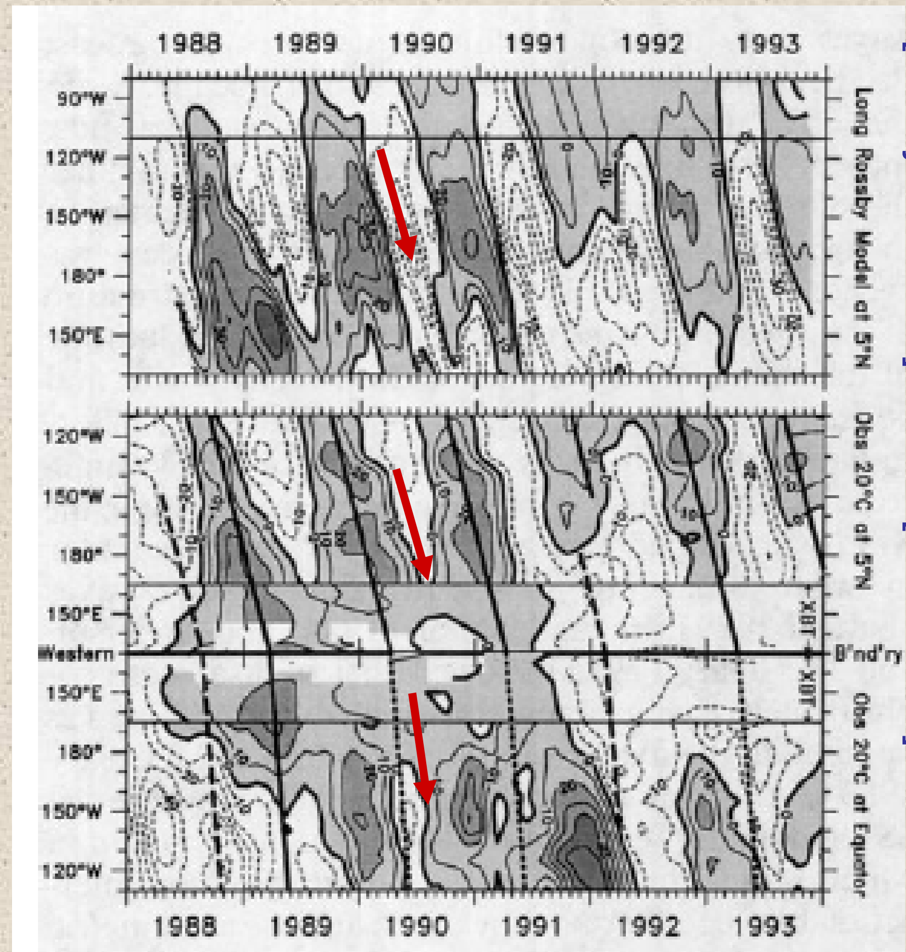
Observations
[4-5 day band]
ENERGY DENSITY (CM^2/CYCLE/HOUR)
10^3
10^2
10^1
CA
AR
OC
JA
HU
CH
FA
KU
MAJ
MAL
KW
EN
GU
WA
KA
HO
MI
S
SOUTH
NORTH
N
LATITUDE (DEGREES)



model
[Wunsch and Gill, 1976]
ENERGY DENSITY (CM^2/CYCLE/HOUR)
10^3
10^2
10^1
10^0
CA
AR
OC
JA
HU
CH
FA
KU
MAJ
BACKGROUND
$(P_2)^2$
WA
MAL
KW
EN
GU
KA
HO
S
SOUTH
NORTH
N
LATITUDE (DEGREES)



The Pacific TAO Array

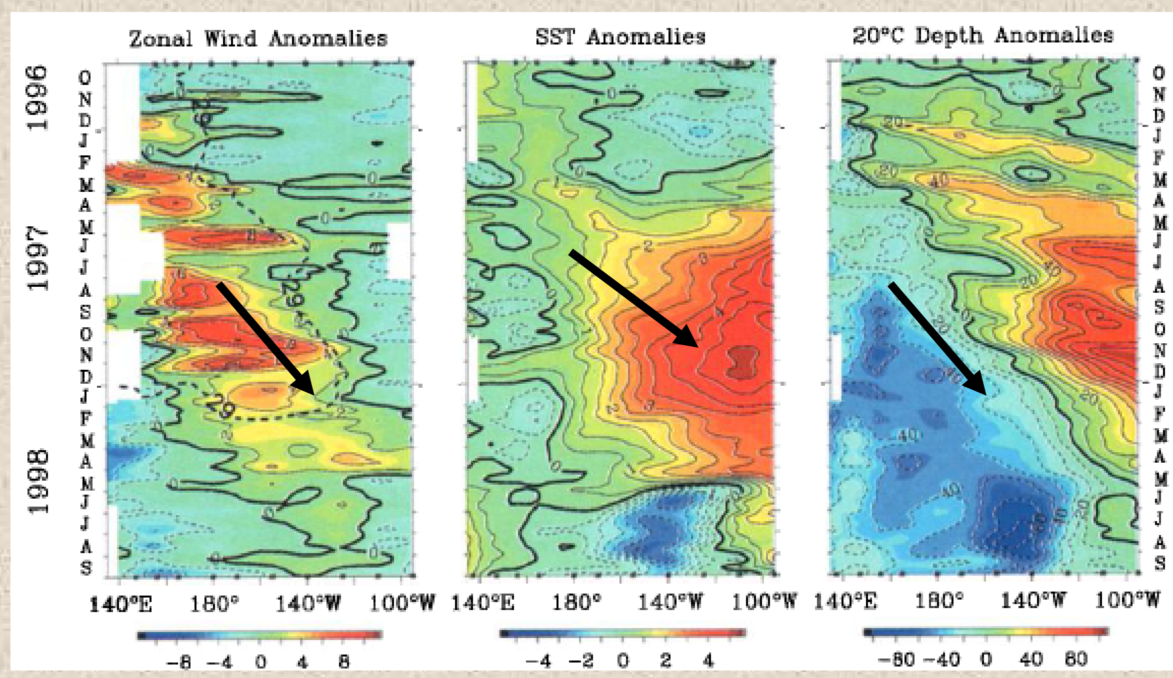


[Rossby wave model]

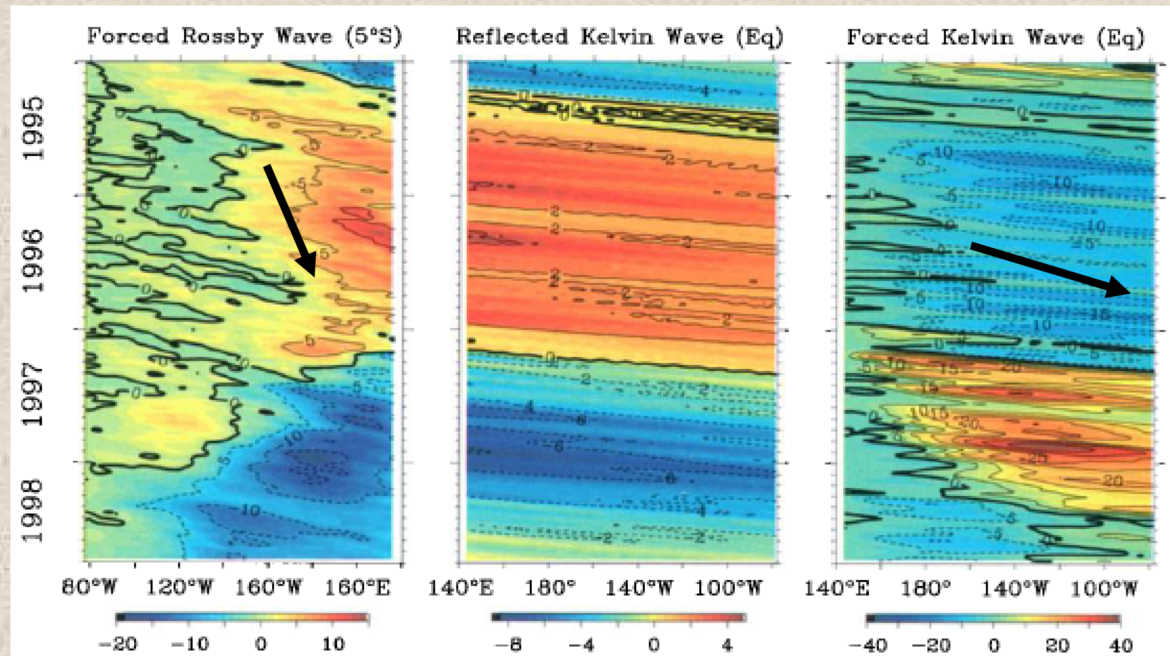
[Observations]

20°C isotherm depth during the 1992-1993 El Niño event

[Kessler and McPhaden, 1995]



Measured quantities from the TAO array during the 1997-1998 El Niño event



Fits of a few equatorial waves to the observations

[McPhaden and Xu, 1999]

Observations of Rossby Waves from Altimetry

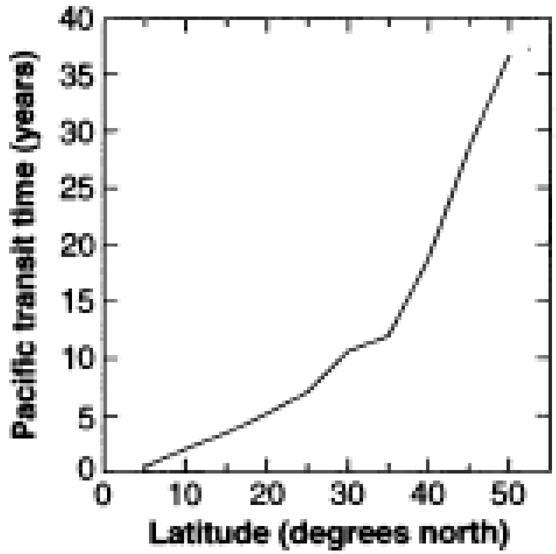


Fig. 1. Latitudinal variation of the time required for baroclinic Rossby waves to cross an ocean basin with the geometry of the North Pacific. These transit times are based on the phase speeds predicted by the standard theory for freely propagating, nondispersive, linear, first-mode baroclinic Rossby waves (*10*).

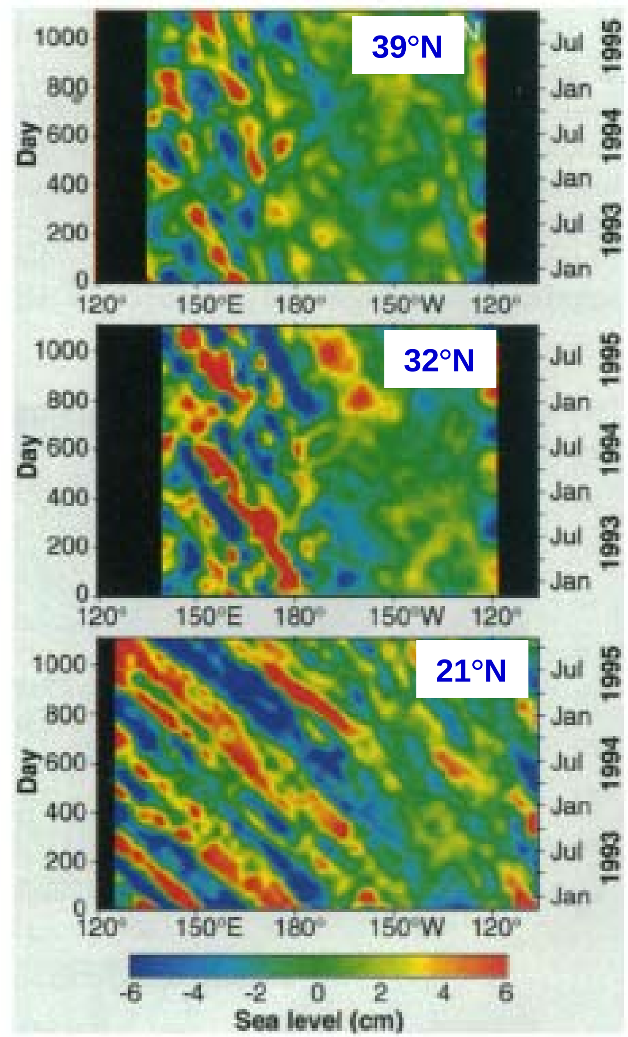
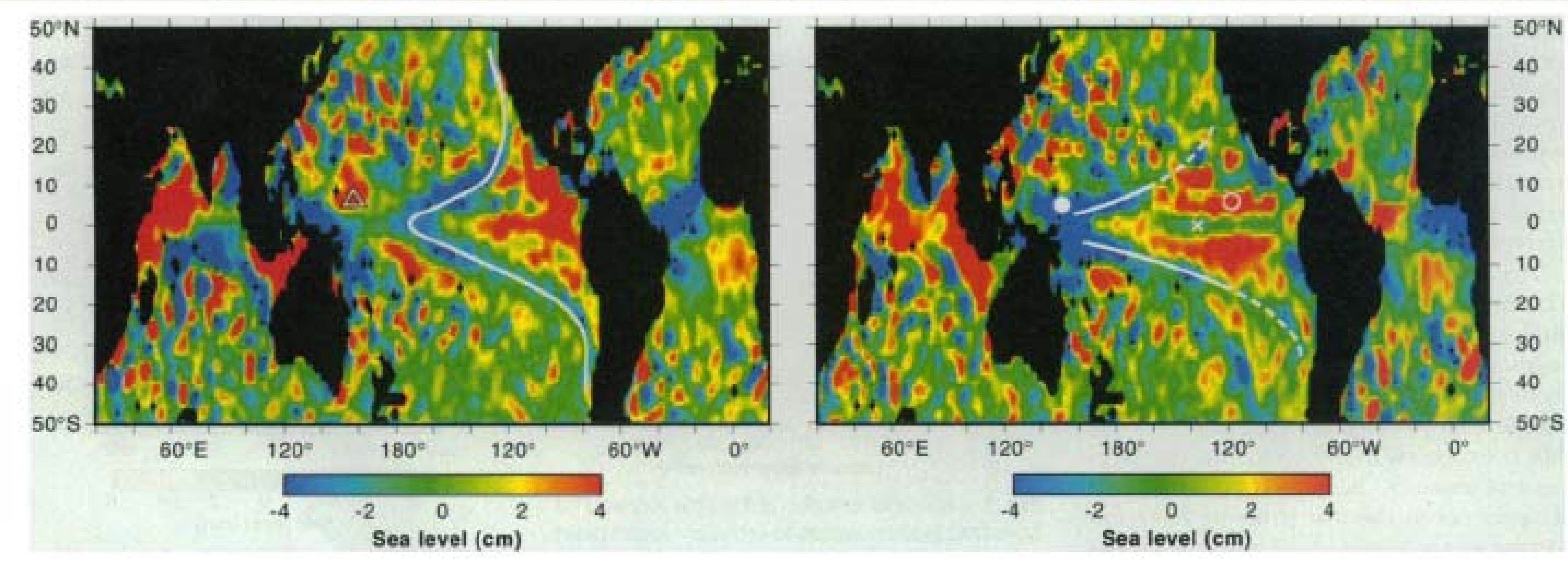


Fig. 2. Time-longitude sections of filtered sea level (*22*) in the Pacific Ocean along 39°, 32°, and 21°N. These examples are representative of extratropical latitudes throughout the world ocean.

[Chelton and Schlax, 1996]

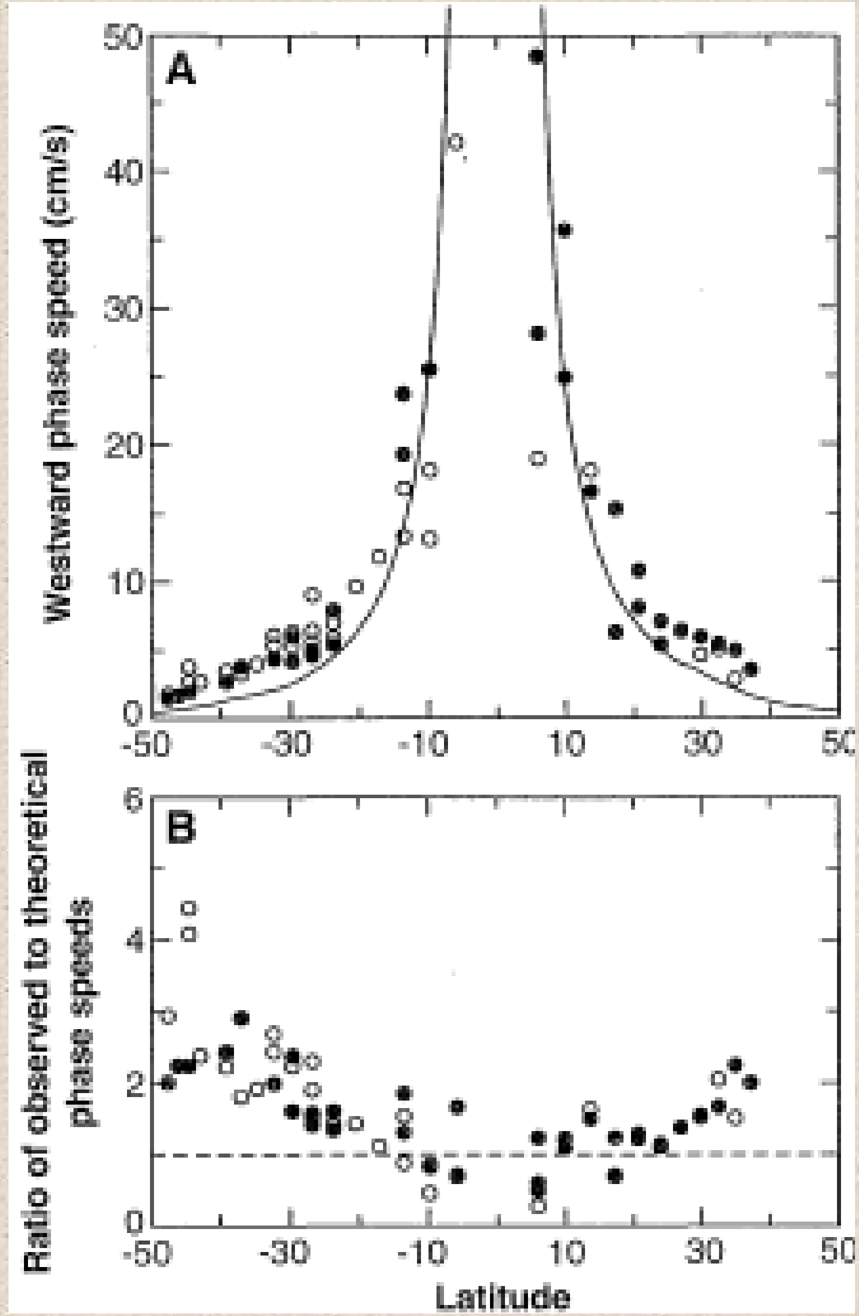
13 April 1993

31 July 1993

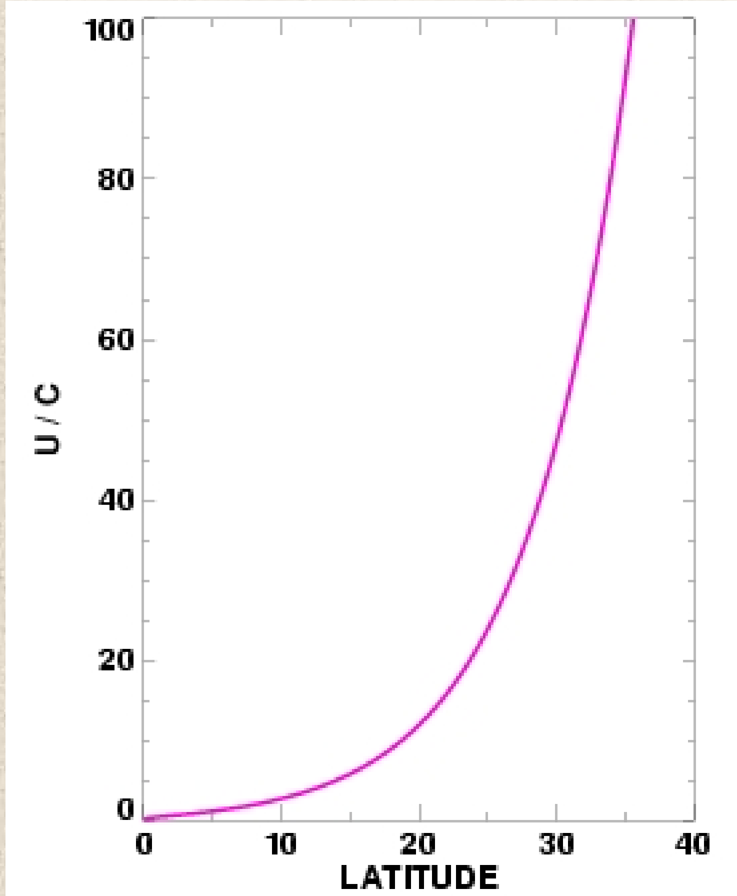


Rossby wave propagation is clearly evident near the Equator; at other latitudes is is more difficult to see, and the sea level field is less spatially coherent.

[Chelton and Schlax, 1996]



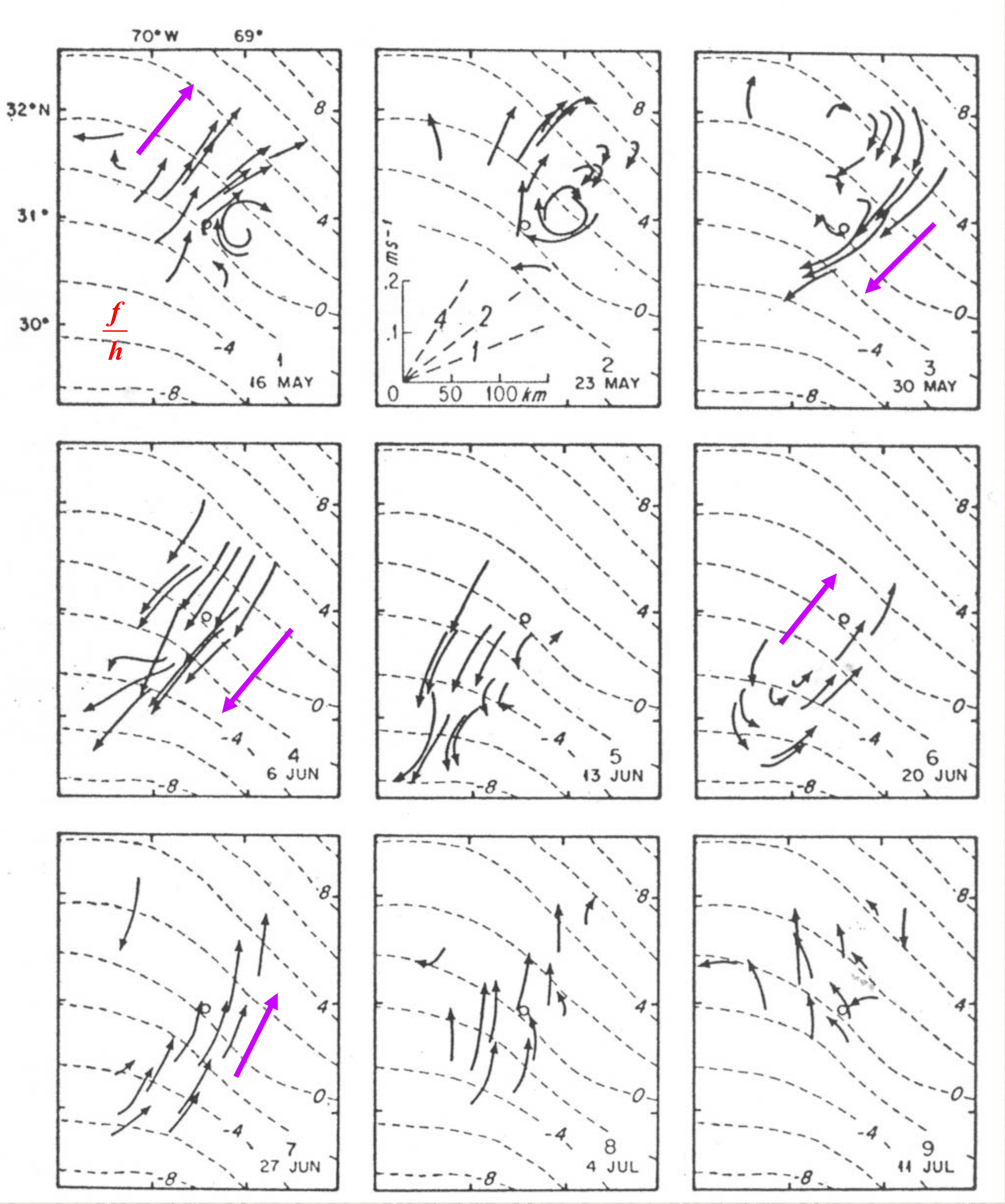
[Chelton and Schlax, 1996]



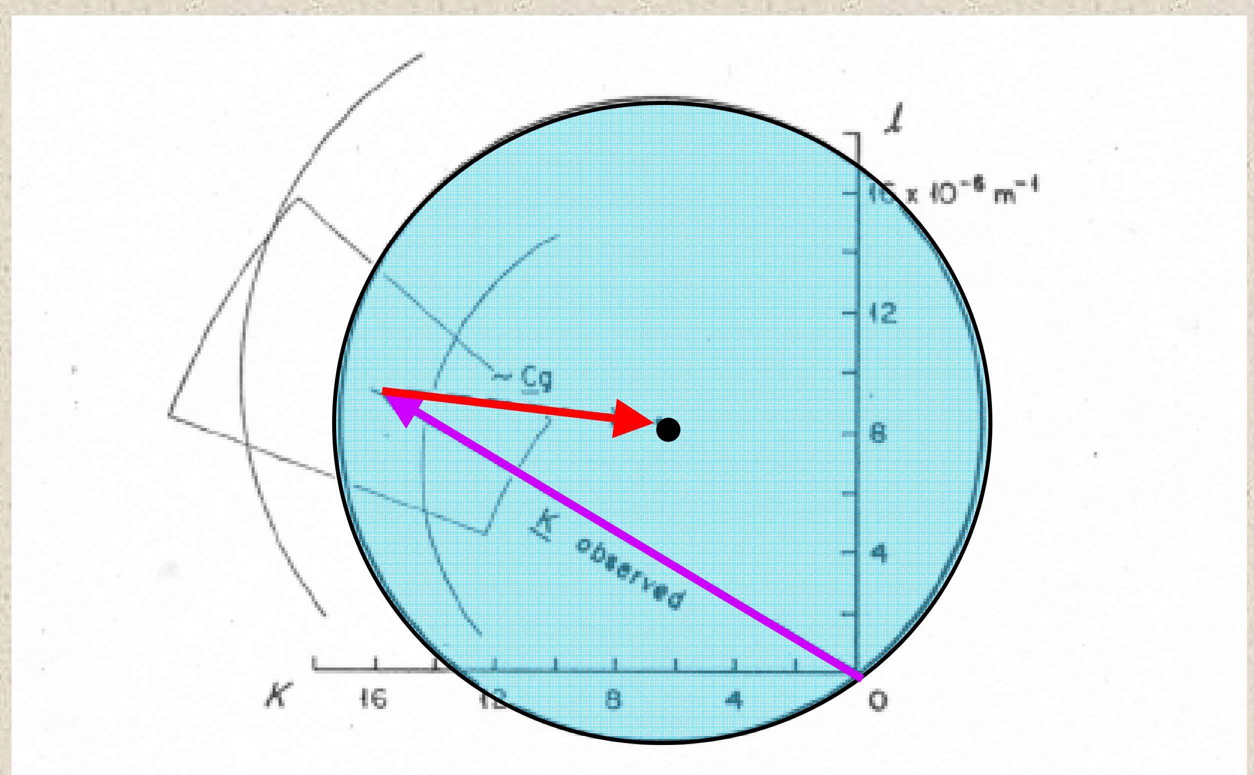
The approximate ratio of particle speed to phase speed as a function of latitude for the Pacific

Result: ***nonlinearity is generally very large except near the Equator!***

The trajectories of 18 acoustically-tracked floats at a depth of 1300 m in the western N. Atlantic during the period 16 May through 11 July, 1979. Note the wavelike motion with particles oscillating on NE/SW trajectories.

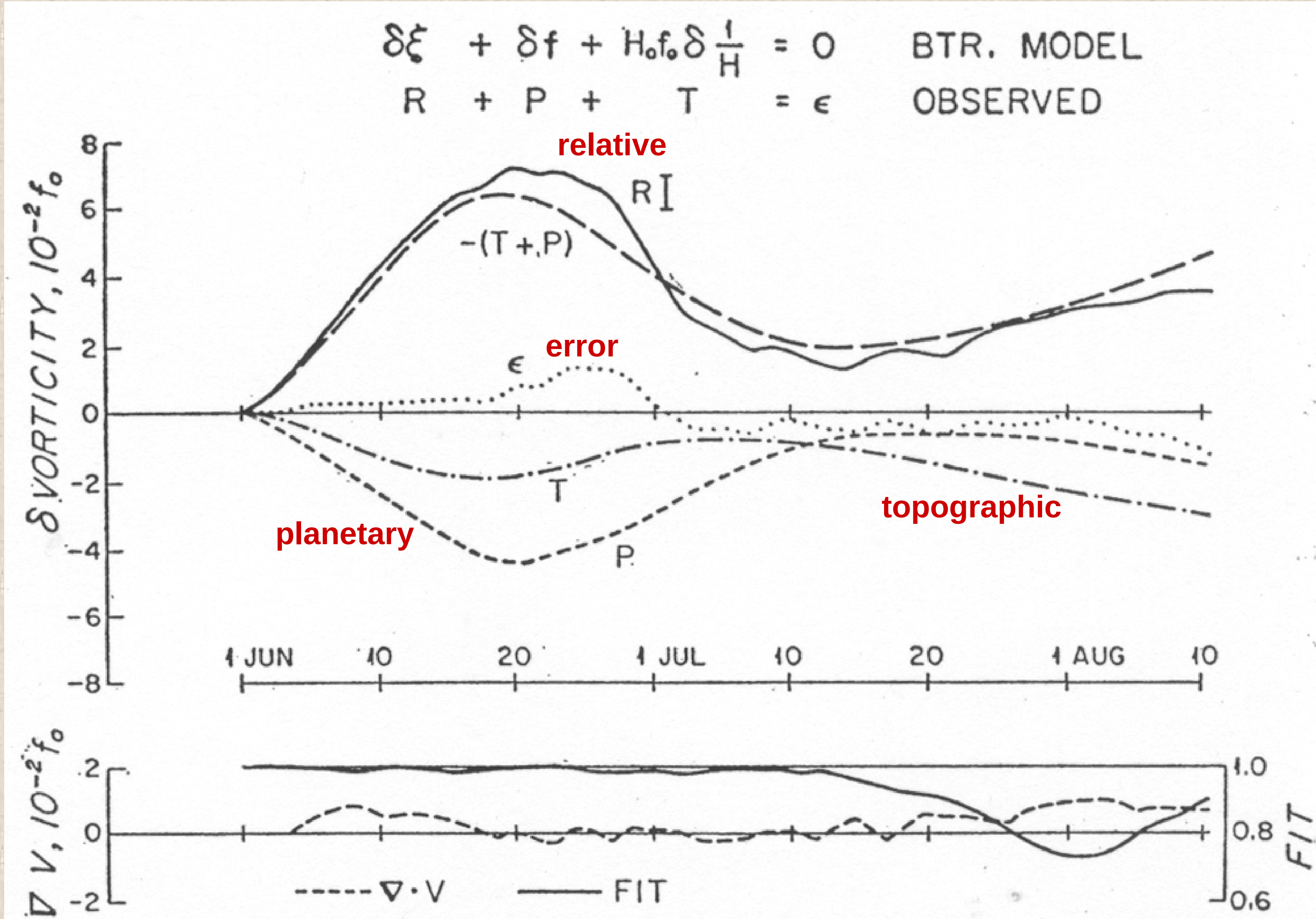


[Price and Rossby, 1982]



The inferred dispersion curve from the Price-Rossby study

[Price and Rossby, 1982]



Estimates of individual terms in the potential vorticity equation from the 1300 m float array, showing approximate balance.

[Price and Rossby, 1982]

Summary

- Rossby waves are the “motions of the second class” originally found by Laplace and studied by many others.

- Rossby’s contribution was to formulate the theoretical problem in Cartesian coordinates, using his β-plane approximation, allowing useful solutions to be found.

- The properties of Rossby waves (their dispersion relation) make them inherently difficult to observe.

- It seems likely that nonlinear effects in Rossby waves are quite strong outside of the near-Equatorial ocean.